

ABCD of 3d $\mathcal{N} = 8$ and 4 Superconformal Field Theories

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ABSTRACT: We argue the equivalence between the infrared conformal field theory of the 3d $\mathcal{N} = 8$ supersymmetric Yang-Mills theories of ABCD ($U(N)$, $SO(2N + 1)$, $Sp(2N)$, $O(2N)$) gauge groups and the ABJ(M) theories of $U(N)_k \times U(\tilde{N})_{-k}$ for $k = 1, 2$. We support this duality by comparing the superconformal index of the IR limit of these super Yang-Mills theories and that of those ABJ(M) models. Especially we find the match between two indices of (mirror dual of) the $\mathcal{N} = 8$ $U(N)$ SYM and of $U(N)_1 \times U(N)_{-1}$ ABJM model. Also we take large N limit of ABCD super Yang-Mills theories with additional fundamental hyper-multiplets and infer the large N limit of $\mathcal{N} = 8$ ABCD theories themselves, finding the expected gravitational duals. With the additional input on finite N , we argue the equivalence of Yang-Mills and ABJ(M) theories for all N . We further explore similar dualities to Chern-Simons matter theories for $\mathcal{N} = 4$ Yang-Mills theories related by mirror symmetry.

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1. Introduction

Recently one have witnessed rapid progress in the understanding of the superconformal field theories (SCFT) associated with M2 branes. The famous example is the theory on N M2 branes on \mathbb{C}^4 , which is realized as the ABJM model which is a $\mathcal{N} = 6$ Chern-Simons matter theory with gauge group $U(N)_k \times U(N)_{-k}$ and Chern-Simons level $k = 1$ [1]. Prior to this development, strongly coupled superconformal field theories(SCFT) have been explored as the infrared (IR) limit of supersymmetric Yang-Mills (SYM) theories in three dimensions [2, 3, 4]. The recent progress has shown especially that the IR limit of the $\mathcal{N} = 8$ $U(N)$ SYM theory flows to the ABJM model with $U(N)_1 \times U(N)_{-1}$. Especially the calculation of the partition functions of two theories matches each other beautifully [5].

An obvious question is whether there is any similar superconformal Chern-Simons matter theory which describes the IR limit, or the infinite coupling limit, of $\mathcal{N} = 8$ super Yang-Mills theory with the general gauge group G . If that is the case, one would like to test the equivalence or duality.

In this work, we explore the IR limit of the $\mathcal{N} = 8$ SYM theories of the gauge group in the ABCD classical groups, say, $U(N)$, $SO(2N + 1)$, $Sp(2N)$, $O(2N)$. We argue that the $O(2N)$ Yang-Mills theory flows to the $U(N)_2 \times U(N)_{-2}$ ABJM model and the $SO(2N + 1)$ Yang-Mills theory flows to $U(N)_2 \times U(N + 1)_{-2}$ ABJ theory [6]. The $\mathcal{N} = 8$ $Sp(2N)$ SYM theory can flow to either ones. The main tool we use is the superconformal index [7, 8, 9].

There are four kinds of orientifold planes $O2^-$, $O2^+$, $\tilde{O}2^+$, $\tilde{O}2^-$ in type IIA string theory and two kinds of orbifold plane $OM2^-$ and $OM2^+$ in the M theory. Our BCD $\mathcal{N} = 8$ supersymmetric Yang-Mills theories originates from the theory on D2 branes near the orientifold planes. Each orientifold plane is made of two $OM2$ planes at the end of 11d line segment in the M-theory. We understand the $\mathcal{N} = 8$ super Yang-Mills theory on D2 branes near the orientifold and the ABJ(M) model near $OM2$ plane. This naturally leads to the correspondence between super Yang-Mills theories and ABJ(M) models.

Our main tool to test the equivalence is the superconformal index. However the usual localization is not directly applicable to the index computation of $\mathcal{N} = 8$ $U(N)$, $O(2N)$, $SO(2N + 1)$, $Sp(2N)$ SYM theories. For the $U(N)$ case, we use the $\mathcal{N} = 4$ mirror dual, or the $\mathcal{N} = 8$ theory with an additional fundamental hypermultiplet, which is similar to the one used for the calculation of the partition function [5]. The match between the index for the $\mathcal{N} = 8$ $U(N)$ SYM theory and that of $U(N)_1 \times U(N)_{-1}$ is tested explicitly in small N in the series expansion. For the $O(2N)$, $SO(2N + 1)$, $Sp(2N)$ cases, the IR SCFTs of the $\mathcal{N} = 8$ SYM theories are not equivalent to the IR SCFT limits of the $\mathcal{N} = 8$ SYM theories with additional fundamental hyper-multiplet.

Thus, we take the roundabout approach to the BCD cases. For small N , we improvise and obtain the indices and show that they are identical to those of ABJ(M) type with $k = 2$. In addition to it, we include arbitrary m fundamental hypermultiplets to super Yang-Mills theories. The corresponding field theoretic index is calculable and we consider its large N limit. This has the gravitational dual as the orbifold of $AdS_4 \times S^7/\mathbb{Z}_2$ of order m whose index has the contributions from the twisted sector. After subtracting off the contribution of the twisted sectors from the field theory index, one obtains \mathbb{Z}_m invariant sectors of $AdS_4 \times S^7/\mathbb{Z}_2$ out of the $\mathcal{N} = 8$ super Yang-Mills theory. By varying m , one can see that large N limit of $\mathcal{N} = 8$ BCD super Yang Mills theory matches with the gravitational calculation on $AdS_4 \times S^7/\mathbb{Z}_2$ space. This in turn can be identified with the index of the ABJ(M) theory with $k = 2$ as some calculations were done for this AdS/CFT correspondence between the gravitational calculation and the ABJ(M) field theoretic calculation has been tested before in Ref. [8]. By further working out the particular orbifold theories for finite N or by considering the Higgsing down to small N theories, one could confirm the proposed dualities for all N .

Once handling $\mathcal{N} = 8$ cases, we can ask the similar question for $\mathcal{N} = 4$ Yang-Mills theories. Here our main focus is the theories arising in the mirror symmetry. As is well known under the mirror symmetry the Coulomb and Higgs branches are interchanged. The $\mathcal{N} = 4$ superconformal field theory is living at the origin of the moduli space where Coulomb and Higgs branch intersecting. One can ask if such SCFT can be described again in terms of supersymmetric Chern-Simons matter theory. We mainly consider $\mathcal{N} = 4$ Yang-Mills type theories which is describable by Hanany-Witten set-up with D3/NS5/D5 branes [10]. As is well known, the mirror symmetry is realized as S-dual transformation in the Hanany-Witten setup. Interestingly the associated superconformal Chern-Simons matter theory can be obtained by T-dual transformation where $\tau \rightarrow \tau + 1$ with τ being axion-dilaton of Type IIB theory. We carry out the index computations, which impressively confirm our proposal on the $\mathcal{N} = 4$ SCFTs.

The contents of our paper is as follows. In Sec.2, we briefly review orientifold planes which made of $OM2$ planes, and present the proposal for the IR limit of the $\mathcal{N} = 8$ supersymmetric Yang-Mills theories in terms of the ABJ(M) models with $k = 1, 2$. In Sec.3, we compute the index of $U(N)_1 \times U(N)_{-1}$ ABJM theory and that of the mirror dual of $\mathcal{N} = 8$ $U(N)$ SYM, and find the perfect agreement. And we carry out the detailed exercise of the large N dual of $\mathcal{N} = 8$ $U(N)$ SYM by working out the field theory index of orbifolded theories of $\mathcal{N} = 8$ $U(N)$ Yang-Mills theory and subtracting the twisted sector contribution in the gravitational side. In Sec. 4, we carry out the similar analysis for $\mathcal{N} = 8$ $O(2N)$, $SO(2N+1)$, $Sp(N)$ theory and show that their gravitational dual is $AdS_4 \times S^7/\mathbb{Z}_2$ in the large N limit. With additional input about the index for small N or by consideration of Higgsing pattern, one can

match Yang-Mills theories to ABJ(M) theories with Chern-Simons level 2. In Sec. 5, we work out various super Chern-Simons matter(SCSM) realizations associated with $\mathcal{N} = 4$ SCFT appearing in the $\mathcal{N} = 4$ mirror symmetry. Again we use the index computation as a main tool to confirm the mirror pair and the associated $\mathcal{N} = 4$ SCFT as SCSM. We also consider the partition function with mass and Fayet-Iliopoulos terms and work out how such parameters are mapped under the duality between $\mathcal{N} = 4$ super-Yang-Mills and SCSM. Various technical details are relegated to several appendices.

2. 3d $\mathcal{N} = 8$ Susy Yang-Mills Theories

The 3d $\mathcal{N} = 8$ susy Yang-Mills theory is obtained from the dimensional reduction of $\mathcal{N} = 1$ 10d super Yang-Mills theory. The bosonic part of the action is given by

$$\frac{1}{g^2} \int d^3x \text{Tr} \left(-\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} \sum_{i=1}^7 (D_\mu \phi_i)^2 + \frac{1}{4} \sum_{i,j=1}^7 ([\phi_i, \phi_j])^2 + \frac{1}{2} \epsilon^{\mu\nu\rho} D_\rho (F_{\mu\nu} \vartheta) \right). \quad (2.1)$$

The last term affects the dynamics as the expectation value of the dual scalar fields of the gauge fields is fixed by the parameter ϑ . Let us first consider the theory with the gauge group $U(N)$. Along the moduli space all A_μ, ϕ_i are commuting so that the gauge group is broken to $U(1)^N$. Along the flat directions, we have

$$\sum_{a=1}^N \left(-\frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2g^2} \sum_{i=1}^7 \partial_\mu \phi_i^a \partial^\mu \phi_i^a + \epsilon^{\mu\nu\rho} F_{\mu\nu}^a \partial_\rho \phi_8^a \right), \quad (2.2)$$

where each element belongs to Cartan torus of $U(N) = U(1)^N$. Here we rescaled compact scalars to have 2π periodicity ($\phi_8^a \sim \phi_8^a + 2\pi$)

$$\vartheta = \sum_{a=1}^N 2g^2 \phi_8^a t^a, \quad (2.3)$$

where t^a s are generators of unbroken $U(1)^N$ gauge group with normalization $\text{Tr}(t^a t^b) = \delta^{ab}$. One can dualize the photon with field equation $F^{a\mu\nu}/g^2 = 2\epsilon^{\mu\nu\rho} \partial_\rho \phi_8^a$ and gets

$$-\frac{1}{2} \sum_a \int d^3x \left(\sum_{i=1}^7 \frac{1}{g^2} (\partial \phi_i^a)^2 + 4g^2 (\partial_\mu \phi_8^a)^2 \right) \quad (2.4)$$

Thus the moduli space for the $\mathcal{N} = 8$ susy Yang-Mills theory with the gauge group $A_{N-1} = U(N)$ is [2]

$$\mathcal{M} = \frac{(\mathbb{R}^7 \times S^1)^N}{S_N} \quad (2.5)$$

where S_N is the permutation group of N elements. In the IR limit, $g \rightarrow \infty$ and the radius of the circle associated with the compact scalar tends to infinite. In this case the moduli space is given by

$$\mathcal{M}_{U(N)} = \frac{\mathbb{C}^{4N}}{S_N}. \quad (2.6)$$

Note that S_N is the Weyl group of $U(N)$ gauge group. More generally for $N = 8$ Yang-Mills theory with gauge group G of rank r , the moduli space is supposed to be

$$\mathcal{M} = \frac{\mathbb{R}^{7r} \times \hat{T}^r}{\mathcal{W}_G} \quad (2.7)$$

where \mathcal{W}_G is the Weyl group of G and \hat{T}^r is the Cartan torus for the dual group [2]. In the IR limit the moduli space is given by

$$\mathcal{M}_G = \frac{\mathbb{C}^{4r}}{\mathcal{W}_G} \quad (2.8)$$

For $B_N = SO(2N+1)$, $C_N = Sp(2N)$, $D_N = O(2N)$ the Weyl group is $\mathbb{Z}_2^N \times S_N$ so that the moduli space for all these gauge groups [4] is

$$\mathcal{M}_{BCD} = \frac{(\mathbb{C}^4/\mathbb{Z}_2)^N}{S_N}. \quad (2.9)$$

Interestingly enough, the vacuum moduli space of $SU(N)$ and $SO(2N)$ are more complicated. For the $SU(N)$ case, the vacuum moduli space is

$$\mathcal{M}_{SU(N)} = \frac{\mathbb{C}^{4(N-1)}}{S_N} \quad (2.10)$$

and the $SO(2N)$ has the Weyl group $\mathcal{W}_{SU(N)} = \mathbb{Z}_2^{N-1} \times S_N$ so that its moduli space is

$$\mathcal{M}_{SO(2N)} = \frac{(\mathbb{C}^4/\mathbb{Z}_2)^N}{S_N} \times \mathbb{Z}_2 \quad (2.11)$$

One notices the vacuum moduli space of the low energy limit or infinite coupling limit of the given $\mathcal{N} = 8$ super Yang-Mills theory becomes simpler for $U(N)$, $SO(2N+1)$, $Sp(2N)$, $O(N)$ gauge groups. Essentially, they allow the interpretation in term of multiple M2 branes exploring either \mathbb{C}^4 or the orbifold $\mathbb{C}^4/\mathbb{Z}_2$.

To understand this view, let us first consider the dynamics of N parallel D2 branes of type IIA string theory in the flat space-time. The transverse 7d space can be either flat \mathbb{R}^7 or $\mathbb{R}^7/\mathbb{Z}_2$ orbifold. In type IIA string theory, there are four kinds of orientifolds: $O2^-$, $O2^+$, $\widetilde{O2}^+$, $\widetilde{O2}^-$ of D2 charge, $-1/8$, $+1/8$, $+1/8$, $+3/8$, respectively. The $\mathcal{N} = 8$ super Yang-Mills theories on N D2 branes in the background

of these orientifold have the gauge group $O(2N), Sp(2N), Sp(2N), SO(2N+1)$, respectively. The gauge group $Sp(2N)$ arises for the two cases with different range of diagonal $Sp(2N)$ matrix ϑ whose eigenvalues denote the positions of M2 branes in the x^{11} direction. Table I below denotes the $O2$ planes which are composed of two $OM2$ planes at two ends of the compact line segment x^{11} of M-theory.

$O2^-$	$OM2^- + OM2^-$	$O(2N)$
$O2^+, \widetilde{O2}^+$	$OM2^- + OM2^+$	$Sp(2N)$
$\widetilde{O2}^-$	$OM2^+ + OM2^+$	$SO(2N+1)$

Table I: $O2$ Plane Made of Two $OM2$ Planes

In the M-theory $\mathbb{C}^4/\mathbb{Z}_2$ orbifold singularity can come in two varieties depending on the presence of discrete torsion. Without discrete torsion, it is called $OM2^-$ plane of M2 charge $-1/16$ and with a discrete torsion, it becomes $OM2^+$ plane with quarter of M2 brane stuck with M2 brane charge $+3/16$ [4]. A M2 brane on all of these orientifold background has the same moduli space $\mathbb{C}^4/\mathbb{Z}_2$. The superconformal field theory on N M2 branes exploring \mathbb{C}^4 is the $\mathcal{N} = 6$ ABJM model of the gauge group $U(N)_1 \times U(N)_1$ with $k = 1$, whose supersymmetry gets enhanced to $\mathcal{N} = 8$ [11]. The theory near $OM2^-$ is that of the gauge group $U(N)_2 \times U(N)_{-2}$ with $k = 2$, whose supersymmetry is also enhanced to $\mathcal{N} = 8$. The theory near $OM2^+$ is that of the gauge group $U(N)_2 \times U(N+1)_{-2}$ with $k = 2$, whose supersymmetry is also enhanced to $\mathcal{N} = 8$ [12].

Let us first note that there exists an important equivalence in physics or duality between the ABJ(M) models [6]:

$$U(N+\ell)_k \times U(N)_{-k} \Longleftrightarrow U(N)_k \times U(N+k-\ell)_{-k}. \quad (2.12)$$

This duality implies the following duality between three $\mathcal{N} = 8$ models for $k = 1$:

$$U(N+1)_1 \times U(N)_{-1} \Longleftrightarrow U(N)_1 \times U(N)_{-1} \Longleftrightarrow U(N)_1 \times U(N+1)_{-1}. \quad (2.13)$$

For $k = 2$, the the following duality holds:

$$U(N+2)_2 \times U(N)_{-2} \Longleftrightarrow U(N)_2 \times U(N)_{-2} \Longleftrightarrow U(N)_2 \times U(N+2)_{-2}. \quad (2.14)$$

One can confirm this duality by calculating the superconformal indices of the next section and comparing them. For example, we find the exact match for $N = 1$ case. The duality

$$U(N+1)_2 \times U(N)_{-2} \Longleftrightarrow U(N)_2 \times U(N+1)_{-2} \quad (2.15)$$

implies that the model of this gauge group is parity even. The parity for this model is given by the usual parity transformation accompanied by the Seiberg-like dualities [12].

Let us now compare the SCFTs limit of the $\mathcal{N} = 8$ supersymmetric Yang-Mills theories with the ABJM and ABJ type superconformal Chern-Simons matter theories with $U(N)_k \times U(N+1)_{-k}$ for $k = 1, 2$. From the brane picture it is obvious now. First of all the vacuum moduli space of the IR limit of the super Yang-Mills theory and that of ABJ(M) should match. The number of supersymmetry should be $\mathcal{N} = 8$. The Table II shows the relation between the IR limit of $\mathcal{N} = 8$ super Yang-Mills theory and the ABJ(M) models.

Type	Super Yang-Mills	Super Chern-Simons
A	$U(N)$ SYM	$U(N)_1 \times U(N)_{-1}$
B	$SO(2N+1)$ SYM	$U(N)_2 \times U(N+1)_{-2}$
C	$Sp(2N)$ SYM	$U(N)_2 \times U(N+1)_{-2}$
C	$Sp(2N)$ SYM	$U(N)_2 \times U(N)_{-2}$
D	$O(2N)$ SYM	$U(N)_2 \times U(N)_{-2}$

Table II: The IR limit of Super Yang-Mills Theories as ABJ(M) Models

As we suspect the equivalence between the IR limit of super Yang-Mills theory and the ABJ(M) models, we expect many quantities of two theories should match. Especially the partition functions on S^3 and the superconformal indices should be identical. In the subsequent sections, we explore this equivalence or duality between these theories by calculating their indices. In the appendix C, we attempt to calculate the partition function and see some matches also.

3. The index of $\mathcal{N} = 8$ $U(N)$ Super Yang-Mills Theory

The 3d $\mathcal{N} = 8$ $U(N)$ super Yang-Mills theory arises as the field theory dynamics on N D2 branes. As the IR limit, the strong coupling limit of the theory on D2 branes becomes the theory on $M2$ branes, which has been identified with the ABJM model with $U(N)_k \times U(N)_{-k}$ and $k = 1$. The index calculation for the ABJM model for all N, k has been done in Ref. [8]. What is new here is the index for the $\mathcal{N} = 8$ super Yang-Mills theory. In this section, we test the equivalence in terms of the index.

The equivalence between the IR limit of super Yang-Mills theory and the ABJM theory has been well tested in terms of the partition function [5]. Similar to the partition function, there is the issue of the divergence which can be avoided similarly by considering the mirror dual which is $\mathcal{N} = 4$ super Yang-Mills theory with one adjoint and one fundamental hyper-multiplets. We also approach the index in the large N limit by the field theory and gravity. By considering the Higgsing pattern, we can also see the consistency of the proposed duality between $\mathcal{N} = 8$ $U(N)$ super Yang-Mills theory and the ABJM theory with Chern-Simons level 1.

3.1 Modules of the Index

Let us start by discussing the general structures of the index for the 3d $\mathcal{N} = 2$ superconformal field theories (SCFTs). Superconformal index for higher supersymmetric theory can be defined using their $\mathcal{N} = 2$ subalgebra. The bosonic subalgebra of the 3-d $\mathcal{N} = 2$ superconformal algebra is $SO(2, 3) \times SO(2)$. There are three Cartan elements denoted by ϵ, j_3 and R which come from three factors $SO(2)_\epsilon \times SO(3)_{j_3} \times SO(2)_R$ in the bosonic subalgebra. One can define the superconformal index for 3-d $\mathcal{N} = 2$ SCFT as follows [7],

$$I = \text{Tr}(-1)^F \exp(-\beta' \{Q, S\}) x^{\epsilon + j_3}, \quad (3.1)$$

where Q is a special supercharge with quantum numbers $\epsilon = \frac{1}{2}, j_3 = -\frac{1}{2}$ and $R = 1$ and $S = Q^\dagger$. They satisfy following anti-commutation relation:

$$\{Q, S\} = \epsilon - R - j_3. \quad (3.2)$$

In the index formula, the trace is taken over gauge-invariant local operators in the SCFT defined on $\mathbb{R}^{1,2}$ or over states in the SCFT on $\mathbb{R} \times S^2$. As is usual for Witten index [13], only BPS states satisfying the bound $\epsilon - R - j_3 = 0$ contributes to the index and the index is independent of β' . If we have additional conserved charges commuting with chosen supercharges (Q, S), we can turn on the associated chemical potentials and the index counts the number of BPS states with the specified quantum number of the conserved charges.

The superconformal index is exactly calculable using localization technique [8],[9]. Following their works, the superconformal index can be written in the following form (for simplicity, we turn off the chemical potentials except for x)

$$I(x) = \sum_{\{s\}} \int d\sigma x^{\epsilon_0} \exp[iS_0] \exp \left[\sum_{p=1}^{\infty} \frac{1}{p} f_{tot}(x^p, e^{ip\sigma_i}) \right]. \quad (3.3)$$

We are considering 3-d $\mathcal{N} \geq 4$ super symmetric Chern-Simons matter theory (SCSM) with gauge group G and hyper-multiplets in R_I (chiral-multiplets in R_I and \bar{R}_I) of G . To take trace over Hilbert-space on S^2 , we impose proper periodic boundary conditions on time direction \mathbb{R} . As a result, the base manifold become $S^1 \times S^2$. For saddle points in localization procedure, we need to turn on monopole fluxes on S^2 and holonomy along S^1 . These configurations of the gauge fields are denoted by $\{s\}$ and $\{\sigma\}$ collectively. Both variables take values in the Cartan subalgebra of G . S_0 denote the classical action for the (monopole+holonomy) configuration on $S^1 \times S^2$. ϵ_0 is called the Casimir energy,

$$\epsilon_0 = \frac{1}{2} \sum_I \sum_{\rho \in R_I} |\rho(s)| - \frac{1}{2} \sum_{\alpha \in G} |\rho(s)|. \quad (3.4)$$

Here $\rho \in R_I$ represent the weights of representation R_I and $\alpha \in G$ denote the roots (weight of adjoint) of G . The f_{tot} can be divided into two parts, $f_{tot} = f_{hyper} + f_{vec}$ where

$$\begin{aligned} f_{hyper}(x, e^{i\sigma}) &= \sum_I \sum_{\rho \in R_I} \frac{x^{1/2}}{1+x} x^{|\rho(s)|} [e^{i\rho(\sigma)} + e^{-i\rho(\sigma)}], \\ f_{vec}(x, e^{i\sigma}) &= - \sum_{\alpha \in G; \alpha \neq 0} e^{i\alpha(\sigma)} x^{|\alpha(s)|}, \end{aligned} \quad (3.5)$$

where we used the fact that the conformal dimensions of fields in hyper-multiplet is canonically $1/2$ for 3-d $\mathcal{N} \geq 3$ theories. The chiral superfields in $\mathcal{N} = 4$ vector-multiplet have conformal dimension 1 and does not contribute to the index. If the action contains the Chern-Simons terms, it gives the nonvanishing contribution,

$$iS_0 = \frac{ik}{4\pi} \int tr(A_0 \wedge dA_0 - \frac{2i}{3} A_0 \wedge A_0 \wedge A_0) = iktr(\sigma s) \quad (3.6)$$

where k is the Chern-Simons level.

For super Yang-Mills, we are taking the IR limit $g_{YM} \rightarrow \infty$ in the above index formula. This makes $S_0 = 0$ in this case. This strategy works for many cases for the computation of the partition function of the Yang-Mills theories with $\mathcal{N} = 4$ supersymmetry. One criterion for the existence of the smooth limit is that $SO(4)_R$ symmetry appearing in the Lagrangian of super Yang-Mills theory becomes a part of the superconformal symmetry. If this holds, the scalars in the vector multiplets and the gauge fields have the IR conformal dimension 1 and becomes irrelevant so that we can drop the kinetic term of the gauge fields [5]. For all of the Yang-Mills theories whose index is computed in this paper, this criterion is satisfied.

3.2 the Index of ABJ(M) Model

For the ABJM (ABJ) model with $U(N)_k \times U(\tilde{N})_{-k}$, let us re-derive the index result [8]. This is supersymmetric Chern-Simons matter theory with the gauge group $U(N)_k \times U(\tilde{N})_{-k}$ with the subscript denoting the Chern-Simons level of the underlying gauge group. The matters consist of two hypermultiplets A, B in (N, \tilde{N}) and $(\bar{N}, \tilde{\bar{N}})$ of G ,

$$\begin{array}{cc} U(N)_k & U(\tilde{N})_{-k} \\ (A_1, B_1) & N \quad \tilde{\bar{N}} \\ (A_2, B_2) & \bar{N} \quad \tilde{N}. \end{array} \quad (3.7)$$

(A_1, A_2) and (B_1, B_2) denote the chiral multiplets in A and B .

Monopole charges are denoted by $\{s\} = \{n_i, \tilde{n}_{\tilde{j}}\}$ and holonomy variables by $\{\sigma\} = \{\lambda_i, \tilde{\lambda}_{\tilde{j}}\}$, where $i = 1, \dots, N$ and $\tilde{j} = 1, \dots, \tilde{N}$. Weights are given by

$$\begin{aligned}\rho &\in (N, \tilde{N}) : \{e_i - \tilde{e}_{\tilde{j}}\}, \\ \rho &\in (\bar{N}, \tilde{N}) : \{-e_i + \tilde{e}_{\tilde{j}}\}, \\ \alpha &\in G : \{e_i - e_j, \tilde{e}_{\tilde{i}} - \tilde{e}_{\tilde{j}}\}.\end{aligned}\tag{3.8}$$

The terms appearing in the general formula (3.3) are given as

$$\begin{aligned}S_0 &= ik \sum_{i=1}^N n_i \lambda_i - ik \sum_{\tilde{i}=1}^{\tilde{N}} \tilde{n}_{\tilde{i}} \tilde{\lambda}_{\tilde{i}}, \\ f_{hyper}(x, e^{i\lambda}, e^{i\tilde{\lambda}}) &= 2 \sum_{i, \tilde{j}} \left(\frac{x^{1/2}}{1+x} x^{|n_i - \tilde{n}_{\tilde{j}}|} e^{i(\lambda_i - \tilde{\lambda}_{\tilde{j}})} + \frac{x^{1/2}}{1+x} x^{|n_i - \tilde{n}_{\tilde{j}}|} e^{i(-\lambda_i + \tilde{\lambda}_{\tilde{j}})} \right), \\ f_{vec}(x, e^{i\lambda}, e^{i\tilde{\lambda}}) &= - \sum_{i \neq j} \left(e^{i(\lambda_i - \lambda_j)} x^{|n_i - n_j|} \right) - \sum_{\tilde{i} \neq \tilde{j}} \left(e^{i(\tilde{\lambda}_{\tilde{i}} - \tilde{\lambda}_{\tilde{j}})} x^{|\tilde{n}_{\tilde{i}} - \tilde{n}_{\tilde{j}}|} \right), \\ \epsilon_0 &= \sum_{i, \tilde{j}} |n_i - \tilde{n}_{\tilde{j}}| - \frac{1}{2} \sum_{i, j=1}^N |n_i - n_j| - \frac{1}{2} \sum_{\tilde{i}, \tilde{j}=1}^{\tilde{N}} |\tilde{n}_{\tilde{i}} - \tilde{n}_{\tilde{j}}|.\end{aligned}\tag{3.9}$$

The index for the ABJ(M) model with $U(N)_k \times U(\tilde{N})_{-k}$ is given by

$$I_{ABJM}(x) = \sum_{\{n\}, \{\tilde{n}\}} \frac{1}{(sym)} \int \frac{d^N \lambda}{(2\pi)^N} \frac{d^{\tilde{N}} \tilde{\lambda}}{(2\pi)^{\tilde{N}}} x^{\epsilon_0} \exp[iS_0] \exp \left[\sum \frac{1}{p} f_{tot}(x^p, e^{ip\lambda_i}, e^{ip\tilde{\lambda}_{\tilde{i}}}) \right].\tag{3.10}$$

Here (sym) is the symmetric factor, that is, the order of Weyl-group for unbroken gauge group in the presence of monopole. For example, $U(2) \times U(2)$ is broken to $U(2) \times U(1)$ due to the monopole $(n_1, n_2) = (1, 1)$, $(\tilde{n}_1, \tilde{n}_2) = (1, 2)$. In this case, the remaining Weyl-group is $S_2 \times S_1$ and the symmetry factor is $|S_2 \times S_1| = 2! \times 1! = 2$.

The above superconformal index for small N and $k = 1$ can be calculated by using Mathematica at any order in x , and becomes

$$\begin{aligned}I_{1_1 \times 1_{-1}}(x) &= 1 + 4x^{1/2} + 10x + 16x^{3/2} + 19x^2 + 20x^{5/2} + 26x^3 + 40x^{7/2} + 49x^4 \\ &\quad + 40x^{9/2} + 26x^5 + 40x^{11/2} + 84x^6 + 100x^{13/2} + 52x^7 + 8x^{15/2} + 64x^8 + 172x^{17/2} \\ &\quad + 150x^9 - 16x^{19/2} - 61x^{10} + 172x^{21/2} + 376x^{11} + 152x^{23/2} - 235x^{12} + \dots\end{aligned}\tag{3.11}$$

$$\begin{aligned}I_{2_1 \times 2_{-1}}(x) &= 1 + 4x^{1/2} + 20x + 56x^{3/2} + 139x^2 + 260x^{5/2} + 436x^3 + 640x^{7/2} + 954x^4 + 1420x^{9/2} \\ &\quad + 2076x^5 + 2720x^{11/2} + 3234x^6 + 3780x^{13/2} + 5012x^7 + 7048x^{15/2} + 8969x^8 + \dots\end{aligned}\tag{3.12}$$

$$\begin{aligned}I_{3_1 \times 3_{-1}}(x) &= 1 + 4x^{1/2} + 20x + 76x^{3/2} + 239x^2 + 644x^{5/2} + 1512x^3 + 3100x^{7/2} + 5734x^4 \\ &\quad + 9856x^{9/2} + 16182x^5 + \dots\end{aligned}\tag{3.13}$$

$$\begin{aligned}I_{4_1 \times 4_{-1}}(x) &= 1 + 4x^{1/2} + 20x + 76x^{3/2} + 274x^2 + 844x^{5/2} + 2392x^3 + 6040x^{7/2} \\ &\quad + 13973x^4 + 29456x^{9/2} + 57756x^5 + \dots\end{aligned}\tag{3.14}$$

3.3 the Index of the $U(N)$ Super Yang-Mills Theory

Here we work out the index for the mirror dual of $\mathcal{N} = 8$ $U(N)$ Yang-Mills theory. The coincidence of the index convincingly reconfirms the equivalence between this ABJM superconformal theory and the IR limit of the $\mathcal{N} = 8$ $U(N)$ Yang-Mills theory. The equivalence between them has been demonstrated analytically by comparing the partition functions of two theories [5]. However, the index computation exhibits far more information of the superconformal theories and we will find quite an agreement.

For the superconformal field theory of a low energy super Yang-Mills theory, we propose here the index formula which is the naive generalization of the one (3.3) given above. At the IR limit, the inverse YM coupling $1/g_{YM}^2$ goes to zero and so the contribution from Yang-Mills action vanishes, or $S_0 = 0$. As commented at the subsection 3.1, this could be justified since the kinetic term of Yang-Mills is irrelevant in the IR limit. The strategy works for many cases for the computation of the partition function of the Yang-Mills theories with $\mathcal{N} = 4$ supersymmetry. But in some of the cases like $\mathcal{N} = 8$ super Yang-Mills theories, naive localization leads to nonsensical results, and so it needs a further consideration. In the index computation, we again consider the mirror dual, the $\mathcal{N} = 4$ $U(N)$ Yang-Mills theory with one adjoint and one fundamental hyper-multiplet. This is the theory on N D2 brane with a single $D6$ brane. In the strong coupling limit the $D6$ brane near the origin is lifted to a smooth geometry in M theory so we are left with N M2 branes. Thus, the relevant quantities for this $\mathcal{N} = 4$ theory is

$$\begin{aligned}
S_0 &= 0, \\
f_{hyper}(x, e^{i\lambda}) &= \sum_{i=1}^N \frac{x^{1/2}}{1+x} x^{|n_i|} (e^{i\lambda_i} + e^{-i\lambda_i}) + \sum_{i,j=1}^N \frac{x^{1/2}}{1+x} x^{|n_i-n_j|} (e^{i(\lambda_i-\lambda_j)} + e^{-i(\lambda_i-\lambda_j)}), \\
f_{vec}(x, e^{i\lambda}) &= - \sum_{i \neq j}^N e^{i(\lambda_i-\lambda_j)} x^{|n_i-n_j|}, \\
\epsilon_0 &= \frac{1}{2} \sum_{i=1}^N |n_i|.
\end{aligned} \tag{3.15}$$

The Casimir energy due to the vector multiplet and the adjoint hyper-multiplet cancel each other. The two hypermultiplets have the canonical conformal dimension $1/2$. However, the adjoint chiral multiplet in the $\mathcal{N} = 4$ vector multiplet has the conformal dimension 1 at the IR limit and so does not contribute to the index. The full index for the IR conformal limit of the $\mathcal{N} = 8$ $U(N)$ super Yang-Mills theory is then given by

$$I_{U(N) \text{ SYM}}(x) = \sum_{\{n\}} \frac{1}{(sym)} \int \frac{d^N \lambda}{(2\pi)^N} x^{\epsilon_0} \exp \left[\sum_{p=1}^{\infty} \frac{1}{p} f_{tot}(x^p, e^{ip\lambda_i}) \right]. \tag{3.16}$$

where the symmetric factor is again the order of the Weyl group of the unbroken gauge group in the presence of magnetic charge.

We are claiming that the index (3.10) of the ABJM model for $U(N)_1 \times U(N)_{-1}$ is identical to that (3.16) of the IR SCFT of the 3d $U(N)$ super Yang-Mills theory. We checked extensively the x expansion of the index for various small values of N and find a perfect agreement between them. This agreement reaffirms the duality between the IR limit of $\mathcal{N} = 8$ super Yang-Mills theory with $U(N)$ gauge group, which is regularized by a single fundamental hyper, and the ABJM model with $U(N)_1 \times U(N)_{-1}$. But it remains as a challenge to show this equivalence in the exact analytic level.

One interesting case is $\mathcal{N} = 8$ $U(1)$ Yang-Mills theory. In the IR limit this describes a single $M2$, which can be described by (supersymmetric) free theory which consists of two hypermultiplets. The 4 complex scalars in the theory correspond to \mathbb{C}^4 where a single $M2$ branes is probing. Thus we expect that the index of $\mathcal{N} = 8$ $U(1)$ super Yang-Mills theory or $U(1)_1 \times U(1)_{-1}$ ABJM theory is the same as the free theory with two hypermultiplets. From index calculation, this can be checked. The superconformal index for the free theory can be easily written as

$$I_{free}(x) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{p} f_{free}(x^n)\right], \quad f_{free}(x) := \frac{4x^{1/2}}{1+x}. \quad (3.17)$$

One can check that it is the same as the superconformal index for $U(1)$ SYM and $U(1)_1 \times U(1)_{-1}$ ABJM by the expansion. All the above computation can be done by turning on the chemical potential. Again the index matches.

To see this equivalence explicitly, we note that the index for the ABJM model with $U(1)_1 \times U(1)_{-1}$ is

$$I_{U(1) \text{ ABJM}} = \sum_{n, \tilde{n} \in \mathbb{Z}} x^{|n-\tilde{n}|} \int \frac{d\lambda}{2\pi} \frac{d\tilde{\lambda}}{2\pi} e^{i(n\lambda - \tilde{n}\tilde{\lambda})} \exp\left[\sum_{p=1}^{\infty} \frac{1}{p} f_{tot}(x^p, e^{ip\lambda}, e^{ip\tilde{\lambda}})\right] \quad (3.18)$$

where

$$f_{tot} = \frac{2x^{1/2}}{1+x} x^{|n-\tilde{n}|} (e^{i(\lambda-\tilde{\lambda})} + e^{-i(\lambda-\tilde{\lambda})}) \quad (3.19)$$

Now we put $r = \lambda - \tilde{\lambda}$, $s = (\lambda + \tilde{\lambda})/2$ with $\int d\lambda d\tilde{\lambda}/(2\pi)^2 = \int dr ds/(2\pi)^2$ with the range for $\lambda, \tilde{\lambda}, r, s$ being $[-\pi, \pi]$. One can integrate over s to get $n = \tilde{n}$ and sum over n of e^{inr} leads to $\delta(r)$ and so the $I_{U(1)ABJM}$ becomes the free theory index I_{free} . So far, we only know the match in the series expansion the explicit equivalence for the index for the super Yang-Mills theory for $U(1)$ which is

$$I_{U(1)SYM} = \sum_{n \in \mathbb{Z}} x^{|n|/2} \int_{-\pi}^{+\pi} \frac{d\lambda}{2\pi} \exp\left(\sum_{p=1}^{\infty} \frac{1}{p} f_{tot}(x^p, e^{ip\lambda})\right) \quad (3.20)$$

where

$$f_{tot} = \frac{x^{1/2}}{1+x} x^{|n|} (e^{i\lambda} + e^{-i\lambda}) + \frac{2x^{1/2}}{1+x} \quad (3.21)$$

3.4 Large N limit and the twisted sector

One of the serious drawback of index computation is that we cannot directly work out the index of $\mathcal{N} = 8$ super-Yang-Mills theory with gauge group G . At the technical level, this is obvious from the index formula in eq. (3.15). The Casimir energy for any monopole operators vanishes because contributions from an adjoint hyper and from $\mathcal{N} = 2$ vector multiplet cancel each other. In addition, there're no CS terms in the SYM theory and thus monopole operators are gauge-invariant by themselves so they need not be combined with charged matters. The index gets divergent as we take the sum over all these energy-zero monopole operators. Thus we cannot compute the index for N D2s directly.

However one can compute the index for the field theory corresponding to N D2s with m D6s for arbitrary nonzero m . The m D6s introduce m fundamental hypermultiplets to $\mathcal{N} = 8$ super-Yang-Mills theory and break supersymmetries to $\mathcal{N} = 4$. Fundamental hypermultiplets give positive Casimir energy to monopole operators and make the superconformal indices of $\mathcal{N} = 4$ theories to be finite. The vacuum moduli space for such theories are composed of the geometric branch from vector and adjoint hyper multiplets and the Higgs branch from fundamental hyper multiplets. We are interested in here the geometric branch. The vector multiplet for $N = 1$ has the charge m Taub-Nut space as moduli space. The explicit metric can be written as

$$ds_{\text{Taub-Nut}}^2 = H d\vec{r} \cdot d\vec{r} + H^{-1} (d\tau + \vec{\omega} \cdot d\vec{r})^2, \text{ where} \\ H = \frac{1}{g^2} + \frac{m}{|\vec{r}|}, \quad \vec{\nabla} H = \vec{\nabla} \times \vec{\omega}. \quad (3.22)$$

Classical moduli space $\mathbb{R}^3 \times S^1$ is 1-loop corrected by integrating out m fundamental hypermultiplets. Here τ denote the dual photon coordinate with peridicity 4π which corresponds to $2\phi_8^a$ in (2.3). The second term in the harmonic function H is due to the 1-loop effect [14, 15]. In the IR limit, where gauge coupling g goes to infinity, the classical part in the harmonic function disappears and the geometry become $\mathbb{C}^2/\mathbb{Z}_m$. Combining the moduli space from adjoint Higgs, total geometric branch become $\mathbb{C}^2 \times (\mathbb{C}^2/\mathbb{Z}_m)$ for $N = 1$. The additional Higgs branch other than geometrical branch gives index from twisted sector, which is discussed below.

The geometric moduli space for the IR limit with general N would be the symmetric product of $\mathbb{C}^2 \times (\mathbb{C}^2/\mathbb{Z}_m)$. Its gravitational dual in the large N limit is given by $AdS_4 \times S^7/\mathbb{Z}_m$. Thus we are in the situation where we know how to compute the field theory index associated with the orbifolded daughter theories in the field theory

and gravity sides while we do not know the field theory index for the parent theory before the orbifolding. Our strategy is by working out the various daughter theories and infer indirectly on the parent theory.

Concretely, the gravity index on $AdS_4 \times S^7/\mathbb{Z}_m$ is given by the \mathbb{Z}_m invariant projection of the index on $AdS_4 \times S^7$ and the twisted sector contribution [16]. Thus if we subtract the twisted sector contribution from the field theory index dual to $AdS_4 \times S^7/\mathbb{Z}_m$, we come to know the \mathbb{Z}_m invariant contribution of the field theory associated with N D2s. By working out the index for arbitrary m we can deduce the field theory index associated with N D2s. The twisted sector contribution comes from D6-D6 states and it is known how to compute in the gravity side [17], [18]. Thus by working on this scheme, one can indirectly work out the index on N D2s in the large N limit, which is index of $\mathcal{N} = 8$ $U(N)$ Yang-Mills theory. We will adopt the similar strategy later for other gauge groups. In this way, we can establish the equivalence between the IR limit of $\mathcal{N} = 8$ $U(N)$ Yang-Mills theory and ABJM theory in Large N limit. By considering the Higgsing pattern one can see that the equivalence should hold for finite N as well.

It is convenient to turn on the chemical potential y for the $U(1)_{diag}$ monopole charge h and define the index as

$$I(x, y) = \text{Tr}(-1)^F x^{\epsilon+j_3} y^h, \quad h = m \sum_{i=1}^N n_i. \quad (3.23)$$

Let us consider $\mathcal{N} = 4$ theory associated with N D2s and m D6s. This is described by adding m fundamental hypermultiplets to $\mathcal{N} = 8$ $U(N)$ Yang-Mills theory. The geometric branch of moduli space for a single D2 is described by $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_m$. One can consider the global rotation

$$(z_1, z_2, z_3, z_4) \rightarrow (z_1, z_2, e^{i\theta} z_3 e^{-i\theta} z_4) \quad (3.24)$$

where this action corresponds to translation in τ -direction in (3.22). Recalling that the τ is the dual photon direction, quantum number under this action can be identified with the monopole charge h . In M-theory picture, D6 branes are mapped to KK monopole ($\mathbb{R}^{1,6} \times \text{Taub-Nut}$) and the 11-th dimensional circle corresponds to the τ coordinate in the Taub-Nut space.

The detailed field theoretic computation of the large N limit is relegated to the appendix B.1. Here we summarize the results of the computation. Let $I_{U(\infty):m}(x, y)$ be the large N index for $U(N)$ Yang-Mills theory with m fundamental hypermultiplets. The salient feature of the index is that it is factorized into

$$I_{U(\infty):m}(x, y) = I_{U(\infty):m}^{(0)}(x) I_{U(\infty):m}^{(+)}(x, y) I_{U(\infty):m}^{(-)}(x, y), \quad (3.25)$$

where $0, +, -$ denotes the zero, positive, negative monopole charge sectors, respectively. From the gravitational side this is mapped to graviton index with zero, positive, negative KK momentum sectors respectively. From the fact that this gives

graviton index, it should be written as Plethystical form

$$I_{U(\infty):m}(x, y) = \exp \sum_{n=1}^{\infty} \frac{1}{n} I_{U(\infty):m:sp}(x^n, y^n), \quad (3.26)$$

where $I_{m:sp}$ denotes the single particle index. From the decomposition of eq. (3.25), the single particle index can be written as

$$\begin{aligned} I_{U(\infty):m:sp} &= I_{U(\infty):m:sp}^{(0)} + I_{U(\infty):m:sp}^{(+)} + I_{U(\infty):m:sp}^{(-)} , \\ &= \sum_{n=-\infty}^{\infty} y^{mn} I_{U(\infty):m:sp}^{(mn)}(x) . \end{aligned} \quad (3.27)$$

Again 0, +, - denotes the zero, positive, negative monopole sector, respectively. For the zero momentum sector, one can explicitly work out the single particle index in field theory and is given in eq. (B.8):

$$I_{U(\infty):m:sp}^{(0)}(x) = \frac{2(x^{1/2} + x - x^{5/2})}{1 - x^{1/2} - x^2 + x^{5/2}} + (m^2 - 1) \frac{x}{(1+x)(1-x^{1/2})^2} . \quad (3.28)$$

Especially for $m = 1$ case, this coincides with the single particle gravity index with zero momentum on $AdS_4 \times S^7$ given in eq. (A.7).

From the gravitational dual perspective, the field theory index can be decomposed into

$$I_{U(\infty):m:sp}(x, y) = I_{S^7/\mathbb{Z}_m:sp}(x, y) + I_{U(\infty):m:twisted:sp}(x, y) . \quad (3.29)$$

In the gravity side the bulk index $I_{S^7/\mathbb{Z}_m:sp}(x, y)$ comes from the single graviton index on $AdS_4 \times S^7$ by keeping the invariant states under the \mathbb{Z}_m orbifolding,

$$I_{S^7/\mathbb{Z}_m:sp}(x, y) = I_{S^7:sp}^{(0)}(x) + \sum_{n \neq 0} y^{mn} I_{S^7:sp}^{(mn)}(x) \quad (3.30)$$

where $I_{S^7:sp}^{(n)}$ denotes the gravity index on $AdS_4 \times S^7$ with n units of KK momentum along 11-th circle. The twisted sector comes from the fixed locus under the \mathbb{Z}_m action, which is $AdS_4 \times S^3$. D6 branes supported on the fixed locus provide this contribution. For a single D6 brane, the worldvolume theory is seven-dimensional maximal supersymmetric theory on $AdS_4 \times S^3$, which consists of a single vector multiplet. Since $AdS_4 \times S^3$ is embedded into the eleven-dimensional $AdS_4 \times S^7$, so that we should use the M-theory picture. However since the world volume theory does not probe the 11-th circle, usual D6 brane picture does make sense since the spectrum should be independent of the radius of the 11-th circle. The spectrum is worked out in [17, 18]. For a single D6, the index of the 7-d world-volume theory is given as

$$I_{sp}^{U(1);AdS_4 \times S^3}(x) = \frac{x}{1-x^2} \left(1 + 2 \frac{x^{1/2}}{1-x^{1/2}}\right) = \frac{x}{(1+x)(1-x^{1/2})^2} . \quad (3.31)$$

For m D6 branes, we have $SU(m)$ gauge groups and we have

$$I_{sp}^{SU(m); AdS_4 \times S^3}(x) = (m^2 - 1) \frac{x}{(1+x)(1-x^{1/2})^2} \quad (3.32)$$

which exactly matches with the second term in eq. (3.28). By assuming the equality between field theory large N index for $m = 1$ with the gravity index on $AdS_4 \times S^7$, which was extensively checked,

$$I_{U(\infty); m=1:sp}(x, y) = I_{S^7:sp}(x, y). \quad (3.33)$$

one can indeed show that twisted sector index defined in (3.29) are exactly same with the index from m D6 branes (3.32). From eq. (A.7), one can see that the graviton index satisfies $I_{S^7:sp}^{(mn)} = x^{\frac{(m-1)|n|}{2}} I_{S^7:sp}^{(n)}(x)$. On the other hand, from eq. (B.14) the large N index satisfies $I_{U(\infty); m:sp}^{(mn)} = x^{\frac{(m-1)|n|}{2}} I_{U(\infty); m=1:sp}^{(n)}$ for $n \neq 0$. Thus, under the assumption (3.33) one can see that $I_{S^7:sp}^{(mn)} = I_{U(\infty); m:sp}^{(mn)}$ for $n \neq 0$. This imply that twisted sector index in (3.29) is

$$\begin{aligned} I_{U(\infty); m:twisted:sp}(x, y) &= \sum_{n=-\infty}^{\infty} (I_{U(\infty); m:sp}^{(mn)} - I_{S^7}^{(mn)}) y^{mn} = I_{U(\infty); m:sp}^{(0)}(x) - I_{S^7}^{(0)}(x), \\ &= (m^2 - 1) \frac{x}{(1+x)(1-x^{1/2})^2}. \end{aligned} \quad (3.34)$$

In the last line we use explicit expression of y^0 part of large N field theory index (3.28) and graviton index on $AdS_4 \times S^7$ (A.7). This exactly match with index from D6s (3.32) as claimed. One peculiar feature is that in the field theory index the twisted sector contribution comes only from the zero-monopole sector. This is sensible since the monopole charge is identified with KK momentum along the 11-th circle and the twisted sector on $AdS_4 \times S^3$ cannot probe the 11-th circle by construction. One can explicitly construct local operators in SYM theory corresponding to twisted sector index. Listing some lowest order example,

$$\begin{aligned} A_I B_J \text{ modulo } F_\Phi &= B_I A_I \sim 0 \quad : \quad (m^2 - 1)x, \\ A_I \Phi_1 B_J, A_I \Phi_2 B_J \text{ modulo } F_\Phi &\sim 0 \quad : \quad 2(m^2 - 1)x^{3/2}, \dots \end{aligned} \quad (3.35)$$

Here F_Φ denote the F -term condition for adjoint chiral superfield Φ in $\mathcal{N} = 4$ vector multiplet. Φ_1, Φ_2 denote chiral superfields in adjoint $\mathcal{N} = 4$ hyper-multiplet and A_I, B_I are chiral superfields in m fundamental hyper-multiplet. Thus overall arguments lead that field theory index defined on N D2s coincide with the gravity index on $AdS^4 \times S^7$ in the large N limit, which is the same as the index of ABJM theory with Chern-Simons level 1. It would be desirable to work out the index on N D2s directly.

3.5 Higgsing of ABJM and super Yang-Mills

In the previous subsection we show that the index of large N limit of $U(N)$ Yang-Mills theory is the same as that ABJM theory with $k = 1$. It's important to extend this equality to finite N . For $U(N)$ case, we resort to the mirror dual of $\mathcal{N} = 8$, $U(N)$ Yang-Mills whose index is the same as that of ABJM for any finite N . This method is not available for other gauge groups. Thus we look for the Higgsing pattern for Yang-Mills theory and ABJM and induce the equivalence of the two theories. Starting from $U(N) \times U(N)$ ABJM one can consider the Higgsing to $U(1)^N \times U(1)^N$ gauge theory. This corresponds to separating all of M2 branes. If the interdistance between any of M2 branes are very large, we expect that we obtain product of free theories in the IR limit.

Let us turn on the vev of one complex scalar Z_4 of $U(N) \times U(N)$ ABJM

$$Z_4 = \bar{Z}^4 = \begin{pmatrix} R_1 & 0 & 0 & \cdots \\ 0 & R_2 & 0 & \cdots \\ 0 & 0 & R_3 & \cdots \end{pmatrix} \quad (3.36)$$

According to [29] et al, all of the offdiagonal components have the mass $\frac{2\pi}{k}(R_i^2 - R_j^2)$. We are taking the limit $R_i, R_j \rightarrow \infty$ with $R_i \neq R_j$. We expect that all of these massive modes are decoupled in such limit. Thus we are left with abelian ABJM and it is sufficient to see what happens to $U(1) \times U(1)$ ABJM with turning on bifundamental vev. $U(1) \times U(1)$ ABJM is given by

$$L = \frac{k}{4\pi}(A_\mu \partial_\nu A_\rho - \tilde{A}_\mu \partial_\nu \tilde{A}_\rho) - D_\mu \bar{Z}^\alpha D^\mu Z_\alpha - i\bar{\Psi}_\alpha D_\mu \Psi^\alpha \quad (3.37)$$

with

$$D_\mu Z_\alpha = \partial_\mu Z_\alpha - i(A_\mu - \tilde{A}_\mu)Z_\alpha \quad (3.38)$$

With $Z_\alpha = R\delta_\alpha^4 + Y_\alpha$, we integrate out $A_\mu - \tilde{A}_\mu$ to obtain

$$L = -D_\mu \bar{Z}^\alpha D^\mu Z_\alpha - i\bar{\Psi}_\alpha D_\mu \Psi^\alpha + \frac{(\frac{k}{4\pi}\epsilon^{\mu\nu\rho}\partial_\nu(A_\rho + \tilde{A}_\rho) + iR(\partial_\mu(\bar{Y}^4 - Y_4) + \bar{\Psi}_\alpha \gamma_\mu \Psi^\alpha + i\partial_\mu \bar{Y}^\alpha Y_\alpha - i\bar{Y}^\alpha \partial_\mu Y_\alpha)^2}{4(R^2 + R(Y_4 + \bar{Y}_4) + Y_\alpha \bar{Y}^\alpha)} \quad (3.39)$$

Replace $A_\rho + \tilde{A}_\rho$ by $R(A_\rho + \tilde{A}_\rho)$ to obtain the standard kinetic term for the gauge field, we obtain

$$L = -\frac{k^2}{32\pi^2}(F + \tilde{F})^2 - \sum_{\alpha=1}^3 \partial_\mu \bar{Z}^\alpha \partial^\mu Z_\alpha - \frac{1}{2}(\partial \frac{\bar{Y}^4 + Y_4}{\sqrt{2}})^2 - i\bar{\Psi}_\alpha \partial_\mu \Psi^\alpha + O(\frac{1}{R}) \quad (3.40)$$

which is $\mathcal{N} = 8U(1)$ Yang-Mills theory. Thus in the limit $R_i, R_j \rightarrow \infty$ with $R_i \neq R_j$ the low energy theory of $U(N) \times U(N)$ ABJM with level k is given by N copies of $\mathcal{N} = 8U(1)$ Yang-Mills theory.

Now consider $\mathcal{N} = 8 U(N)$ super Yang-Mills theory and $U(N)_1 \times U(N)_{-1}$ ABJM theory. They have the same moduli space and the same large N limit. Also Higgsing pattern is consistent. From the Higgsing from $U(N)$ Yang-Mills, we can obtain $U(1)^N$ Yang-Mills. In the ABJM side, we obtain the same $U(1)^N$ Yang-Mills theory after the Higgsing. After the Higgsing, they have the same index for a trivial reason. In BCD case, this kind of argument is effective in telling which Yang-Mills theory should be mapped to which ABJ type theory.

4. BCD super Yang-Mills and $k = 2$ ABJ(M) Models

We are proposing the dualities between the IR limit of $\mathcal{N} = 8$ super Yang-Mills theories and the ABJ(M) models. In this section we are interested in the following duality

$$\begin{aligned} O(2N) \text{ SYM} &\implies U(N)_2 \times U(N)_{-2} && \text{ABJM} \\ SO(2N+1) \text{ SYM} &\implies U(N)_2 \times U(N+1)_{-2} && \text{ABJ} \\ Sp(2N) \text{ SYM} &\implies U(N)_2 \times U(N)_{-2} && \text{ABJM} \\ &\implies U(N)_2 \times U(N+1)_{-2} && \text{ABJ} \end{aligned} \quad (4.1)$$

In this section, we want to test the proposal in Sec.2 for the correspondence between the infrared limit of the $\mathcal{N} = 8$ supersymmetric Yang-Mills theories of BCD type gauge group and the ABJ(M) models, which is summarized in Table II. The main tool is to compare the indices of ABJ(M) models and those of super Yang-Mills. Unlike the previous section for $U(N)$ super Yang-Mills theory, there is no clear regularization process for the BCD case and we improvise the index for small N , and use the large N limit of the field theory and gravity calculation to infer the index. For $N = 2$ the duality is proposed in [19].

Let us first calculate the indices (3.10) of the ABJ(M) model of $U(N)_k \otimes U(\tilde{N})_{-k}$ for $k = 2$ for small N :

$$I_{ABJM}[1_2, 1_{-2}] = 1 + 10x + 19x^2 + 26x^3 + 49x^4 + 26x^5 + 84x^6 + 52x^7 + \dots \quad (4.2)$$

$$I_{ABJM}[1_2, 2_{-2}] = 1 + 10x + 20x^2 + 20x^3 + 65x^4 + 10x^5 + 55x^6 + 190x^7 + \dots \quad (4.3)$$

$$I_{ABJM}[2_2, 2_{-2}] = 1 + 10x + 75x^2 + 220x^3 + 475x^4 + 1060x^5 + 1665x^6 + \dots \quad (4.4)$$

$$I_{ABJM}[2_2, 3_{-2}] = 1 + 10x + 75x^2 + 230x^3 + 449x^4 + 1026x^5 + 1990x^6 + \dots \quad (4.5)$$

$$I_{ABJM}[3_2, 3_{-2}] = 1 + 10x + 75x^2 + 450x^3 + 1595x^4 + 4230x^5 + \dots \quad (4.6)$$

$$I_{ABJM}[3_2, 4_{-2}] = 1 + 10x + 75x^2 + 450x^3 + 1650x^4 + 4240x^5 + \dots \quad (4.7)$$

4.1 Index of $\mathcal{N} = 8$ BCD Yang-Mills Theory for small rank gauge groups

We calculate the indices for $O(2N)$, $SO(2N+1)$, $Sp(2N)$ for small N . Before we consider these cases, let us consider the index (3.16) for the super Yang-Mills theory

for $SU(N)$ gauge group. Note that

$$U(N) = \frac{SU(N) \otimes U(1)}{\mathbb{Z}_N} \quad (4.8)$$

Thus we propose the index for $SU(N)/\mathbb{Z}_N$ $\mathcal{N} = 8$ SYM theory to be

$$I_{SU(N)/\mathbb{Z}_N} = \frac{I_{U(N)}}{I_{U(1)}} \quad (4.9)$$

As there are only adjoint matter fields in the \mathcal{N} theory, the above index is really the index of $SU(N)$ $\mathcal{N} = 8$ SYM theory. For example

$$I_{SU(2)/\mathbb{Z}_2} = 1 + 10x + 20x^2 + 20x^3 + 65x^4 + 10x^5 + 55x^6 + 190x^7 + \dots \quad (4.10)$$

$$I_{SU(3)/\mathbb{Z}_3} = 1 + 10x + 20x^{3/2} + 40x^2 + 104x^{5/2} + 160x^3 + 361x^4 + 516x^{9/2} + \dots \quad (4.11)$$

$$I_{SU(4)/\mathbb{Z}_4} = 1 + 10x + 20x^{3/2} + 75x^2 + 164x^{5/2} + 450x^3 + 780x^{7/2} + 1595x^4 + 2500x^{9/2} + 4230x^5 + \dots \quad (4.12)$$

and so on.

Let us now consider the simplest case, the $\mathcal{N} = 8$ super Yang-Mills theory with $O(2) = SO(2) \otimes \mathbb{Z}_2 = U(1) \otimes \mathbb{Z}_2$, where \mathbb{Z}_2 acts on a complex scalar by its complex conjugation and the sign change for an adjoint scalar. The fractional power in the index for $U(1)$ gauge group denotes the odd number of fields. Thus we can regard the $O(2)$ index to be that of $U(1)$ index without fractional power:

$$I_{O(2)}(x) = 1 + 10x + 19x^2 + 26x^3 + 49x^4 + 26x^5 + 84x^6 + 52x^7 + 64x^8 + 150x^9 - 61x^{10} + 376x^{11} - 235x^{12} + \dots \quad (4.13)$$

This matches the index (4.2) of ABJM with $U(1)_2 \times U(1)_{-2}$ exactly.

The index for $O(4)$ needs a bit more consideration. First of all $SO(4) = SU(2) \times SU(2)/\mathbb{Z}_2$ for the $4d$ real vector representation. For the adjoint representation $SO(4) = SO(3) \times SO(3)$. Thus $O(4) = SO(3) \times SO(3) \times \mathbb{Z}_2$. The vacuum moduli space of the IR dynamics of $O(4)$ super Yang-Mills theory is

$$\mathcal{M}_{O(4)} = (\mathbb{C}^4/\mathbb{Z}_2)^2/\mathbb{Z}_2 = (\mathcal{M}_{SO(3)})^2/\mathbb{Z}_2 \quad (4.14)$$

Thus, we expect that $O(4)$ index to be that of two particle index of $SO(3)$:

$$I_{O(4) \text{ SYM}} = \frac{1}{2} \left\{ I_{SO(3) \text{ SYM}}(x^2) + \left[I_{SO(3) \text{ SYM}}(x) \right]^2 \right\} \quad (4.15)$$

which matches exactly that (4.4) of $U(2)_2 \times U(2)_{-2}$ ABJM model.

The index for the $O(6) = SO(6) \times \mathbb{Z}_2$ super Yang-Mills theory can be obtained from that (4.12) for $SO(6) = SU(4)$ by dropping the fractional power similar to the $O(2)$ case:

$$I_{O(6)} = 1 + 10x + 75x^2 + 450x^3 + 1595x^4 + 4230x^5 + \dots \quad (4.16)$$

which matches the index (4.6) of ABJM model for $U(2)_2 \times U(3)_{-2}$.

The index for the $SO(3) = Sp(2)$ case is the index for the $SU(2)$ super Yang-Mills theory, which is $I_{U(2)}/I_{U(1)}$. The direct calculation (4.10) matches the index (4.3) of ABJM model for $U(1)_2 \times U(2)_{-2}$. On the other hand we know that for $SU(2) = Sp(2)$ there could be another superconformal field theory where this theory flows. There could be subtleties in taking the IR limit. In the low energy limit, the gauge group is reduced to $U(1)$ with seven scalars ϕ_i and ϕ_8 , dual scalar to a photon. The moduli space with the finite coupling is given by $\mathcal{M} = \frac{\mathbb{R}^7 \times S^1}{\mathbb{Z}_2}$ as a special case of (2.7). It has two singularities at $\phi_i = \phi_8 = 0$ and $\phi_i = 0, \phi_8 = \pi$. In the infinite coupling limit, the theory at two orbifold singularities has the moduli space of $\mathbb{R}^8/\mathbb{Z}_2$. It is argued in [2], the theory at $\phi_i = \phi_8 = 0$ leads to interacting conformal theory while the theory at $\phi_i = 0, \phi_8 = \pi$ is a free field theory with a gauged \mathbb{Z}_2 symmetry. Note that ϕ_8 corresponds to the position of M2 brane in the 11-th circle and the different value of ϕ_8 implies the different $OM2$ -plane in the M-theory setting. Note that for the theory defined at $\phi_8 = \pi$, the \mathbb{Z}_2 flips the sign of the scalar field ϕ_i, ϕ_8 so that only the operators of the even ϕ_i, ϕ_8 will survive. Thus from the free field theory index we have to remove the operators having half-integer powers. The resulting index is the same as that of $U(1)_2 \times U(1)_{-2}$ ABJM theory. The other superconformal field theory living at $\phi_i = \phi_8 = 0$ can be identified with $U(1)_2 \times U(2)_{-2}$ ABJ theory.

To summarize, we explicitly checked the indices of the following cases:

SYM	Index	ABJM
$O(2)$	$I_{U(1)} _{no\text{fraction}}$	$U(1)_2 \times U(1)_{-2}$
$SO(3)$	$I_{SU(2)}$	$U(1)_2 \times U(2)_{-2}$
$O(4)$	$(I_{SU(2)}(x^2) + [I_{SU(2)}(x)]^2)/2$	$U(2)_2 \times U(2)_{-2}$
$O(6)$	$I_{SU(4)} _{no\text{fraction}}$	$U(3)_2 \times U(3)_{-2}$
$Sp(2)$		$U(1)_2 \times U(1)_{-2}$
		$U(1)_2 \times U(2)_{-2}$

(4.17)

It would be desirable to define the indices for these BCD class for larger N . Adding one fundamental hyper-multiplet does not seem to work unlike the $U(N)$ case. But this leads to the \mathbb{Z}_2 orbifold theories we will consider in the next subsection.

4.2 \mathbb{Z}_2 Orbifolded Theories

Obviously the above computation can be done only for small ranks of the gauge group. Our roundabout way of the computation consists of several steps. The first step is to work out \mathbb{Z}_2 orbifolded theories of $\mathcal{N} = 8, O(2N)/Sp(2N)$ theories and match to a suitable Chern-Simons dual for any N . The second step is to work out \mathbb{Z}_m orbifolded theories of $\mathcal{N} = 8, BCD$ theories and show that these theories are dual to the gravity theory on $AdS_4 \times S^7/\mathbb{Z}_2$ in the large N limit after taking account

of the twisted sector contribution to the index. From the first and the second step, one can argue the equivalence between $\mathcal{N} = 8, O(2N)/Sp(2N)$ and ABJ(M) theory with $k = 2$ for any finite N . In this calculation, $Sp(2N)$ naturally matches to $U(N)_2 \times U(N+1)_{-2}$. For $SO(2N+1)$ and another branch of $Sp(2N)$, we rest on the Higgsing pattern to check the consistency of the proposed dualities.

Let us look for the theory with $2N$ D2-branes with $2m$ D6-branes and carry out the orientifold projection. This gives $O(2N)/SO(2N+1)/Sp(2N)$ gauge group for D2s. Consider the brane system $2N$ D2s (012) and $2m$ D6s (012345). Lifting to M-theory, D2s become M2-branes probing $\mathbb{C}^2/\mathbb{Z}_{2m} \times \mathbb{C}^2$. Matter fields in the D2/D6 system are

$$\begin{array}{ccc}
& U(2N) & U(2m) \\
\text{Hyper} & Adj & \\
\text{Hyper} & & Adj \\
\text{Hyper} & \frac{2N}{2N} & \frac{2m}{2m}
\end{array} \tag{4.18}$$

$U(2N)$ is the gauge symmetry on D2-branes and $U(2m)$ is the gauge symmetry on D6s, which is the global symmetry of the D2-brane world-volume theory. After introducing $O2^-$ (012), the geometry probed by M2-branes becomes $(\mathbb{C}^2/\mathbb{Z}_{2m} \times \mathbb{C}^2)/\mathbb{Z}_2$. Matter fields in the system are projected into

$$\begin{array}{ccc}
& O(2N) & Sp(2m) \\
\text{Hyper} & Adj & \\
\text{Hyper} & & Adj \\
(\text{real})\text{Hyper} & 2N & 2m
\end{array} \tag{4.19}$$

Combining each pair of $2m$ (real) hyper-multiplets, one can make m hypermultiplets. The world-volume theory of D2s is given by $O(2N)$ $\mathcal{N} = 4$ SYM with hypermultiplets, one in the adjoint and m in $2N$ vector representation. In the IR limit, the theory becomes N M2-branes' world-volume theory on $(\mathbb{C}_1^2/\mathbb{Z}_{2m} \times \mathbb{C}_2^2)/\mathbb{Z}_2$. The action of the discrete quotient on $\mathbb{C}_1^2 \times \mathbb{C}_2^2$ is given by

$$\begin{aligned}
\alpha &= \exp\left(\frac{4\pi i J_3}{2m}\right) \otimes \mathbb{I}, \\
\beta &= \exp(\pi i J_2) \otimes (-\mathbb{I}) = \exp(\pi i J_2) \otimes \exp(2\pi i J'_3).
\end{aligned} \tag{4.20}$$

Here α, β is the generators of $\mathbb{Z}_{2m}, \mathbb{Z}_2$ respectively. $\{J_i, J'_i\}$ are generators of $SU(2), SU(2)'$ which act on \mathbb{C}_1^2 and \mathbb{C}_2^2 respectively. Focusing only on the first \mathbb{C}_1^2 factor in \mathbb{C}^4 , (α, β) generate dihedral group D_m ($4m$ elements) action on \mathbb{C}_1^2 .

On the other hand, related world-volume theory of M2s on $(\mathbb{C}^2/\mathbb{Z}_{2m} \times \mathbb{C}^2)/\mathbb{Z}_2$ can be obtained from the Hannay-Witten setup in Type IIB theory with N D3 branes,

$2m$ NS5-branes and an $(1, 2)$ -brane[27]. But in this case, the \mathbb{Z}_2 action is different from the $O(2N)$ SYM. The generator of the \mathbb{Z}_2 , say $\tilde{\beta}$, act on \mathbb{C}^4 as

$$\tilde{\beta} = \exp\left(\frac{4\pi i J_3}{4m}\right) \otimes \exp(2\pi i J'_3). \quad (4.21)$$

In this case, $(\alpha, \tilde{\beta})$ form cyclic group $\mathbb{Z}_{2 \times 2m} = \mathbb{Z}_{4m}$ action on the first \mathbb{C}_1^2 . $m = 1$ is a special case, when $D_m = \mathbb{Z}_{4m}$ and group action generated by (α, β) are equivalent to action generated by $(\alpha, \tilde{\beta})$ up to some basis change in \mathbb{C}^4 . Note that 2 NS5-branes with an $(1, 2)$ -brane gives rise to $U(N)_2 \times U(N)_0 \times U(N)_{-2}$ $\mathcal{N} = 4$ SCSM with bifundamental matters $(N, \bar{N}, 1), (1, N, \bar{N}), (\bar{N}, 1, N)$. Thus we suggest

$$\begin{aligned} &\mathcal{N} = 4 \ O(2N) \text{ SYM with one hyper in adjoint and one in } 2N \text{ vector} \\ \implies_{IR} &\mathcal{N} = 4 \text{ SCSM with gauge group } U(N)_2 \times U(N)_{-2} \times U(N)_0. \end{aligned} \quad (4.22)$$

We confirm the duality by comparing superconformal index. One subtlety in the calculation is that the SYM gauge group is $O(2N)$ but not $SO(2N)$. After taking this subtlety [21],[22], the index of the SYM for $N = 1$ as an example, is given by

$$I(x) = 1 + 7x + 4x^{3/2} + 16x^2 + 4x^{5/2} + 21x^3 + 8x^{7/2} + 40x^4 + 28x^{9/2} + 34x^5 + \dots \quad (4.23)$$

This matches the index of SCSM with $U(1)_2 \times U(1)_{-2} \times U(1)_0$.

Instead of considering $O2^-$ one can consider $\tilde{O}2^+$. In this case we have $Sp(2N) \times SO(2m)$ gauge group for N D2 and m D6s. By the similar logic, one can see that one can compare $Sp(2N)$ SYM to the Chern-Simons type theory. In this case the IR limit of the $Sp(2N)$ SYM with one fundamental hyper-multiplet is equivalent to the $\mathcal{N} = 4$ SCSM with gauge group $U(N)_2 \times U(N+1)_{-2} \times U(N)_0$:

$$\begin{aligned} &\mathcal{N} = 4 \ Sp(2N) \text{ SYM with one hyper in adjoint and one in } 2N \text{ vector} \\ \implies_{IR} &\mathcal{N} = 4 \text{ SCSM with gauge group } U(N)_2 \times U(N+1)_{-2} \times U(N)_0. \end{aligned} \quad (4.24)$$

Note that in the index computation of $Sp(2N)$, we implicitly choose the value of $\theta = 0$ in (2.1). We do not know how to implement the other value in the index computation. We also check (4.24) by the index. Again as an example, the index of $\mathcal{N} = 4$ $Sp(2)$ SYM and $U(1)_2 \times U(2)_{-2} \times U(1)_0$ SCSM coincide with each other, given by

$$I(x) = 1 + 5x + 8x^{3/2} + 9x^2 + 12x^{5/2} + 16x^3 + 4x^{7/2} + 29x^4 + 56x^{9/2} + O(x^5).$$

On the other hand, we attempt to find similar theory for $SO(2N+1)$ theory, but we do not succeed. This would be an interesting problem to find such theory.

4.3 Large N limit and Twisted Sector

Let us first explore the large N limit on the superconformal index for $O(2N)$ $\mathcal{N} = 4$ SYM with m hyper-multiplets in $2N$. The difference between $O(2N)$ and $SO(2N)$ gauge group come from the ‘baryonic’ operators K and L of the form

$$K = \epsilon_{i_1 i_2 \dots i_{2N}} A^{i_1} A^{i_2} \dots A^{i_{2N}}. \quad (4.25)$$

$$L = \epsilon_{i_1 i_2 \dots i_{2N}} \Phi^{i_1 i_2} \Phi^{i_3 i_4} \dots \Phi^{i_{2N-1} i_{2N}} \quad (4.26)$$

A^i denote the scalar in a hyper-multiplet in $2N$, Φ^{ij} in the adjoint representation. These ‘baryonic’ operators are invariant under $SO(2N)$ but variant under $O(2N)$. Index contributions from these operators start at $o(x^{N/2})$ which is negligible in the large N limit. Thus, in the large N we don’t need to distinguish $O(2N)$ gauge group from $SO(2N)$.

Superconformal index formula for the $SO(2N)$ SYM theory and its large N limit are explicitly presented in the appendix B.2. Large N superconformal index from zero charge monopole, $I_{O(\infty):m}^{(0)}$, can be expressed in terms of plethystic expansion as

$$I_{O(\infty):m}^{(0)}(x) = \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} I_{O(\infty):m:sp}^{(0)}(x^n) \right],$$

where the single particle index is given in eq. (B.17) as

$$I_{O(\infty):m:sp}^{(0)}(x) = \frac{x(3 + 2x^{1/2} + 2x - 2x^{5/2} - x^3)}{(1 - x^2)^2} + (2m^2 + m) \frac{x}{(1 - x)^2} + (2m^2 - m - 1) \frac{2x^{3/2}}{(1 - x)^2(1 + x)}. \quad (4.27)$$

On the other hand, the corresponding gravity bulk index is obtained in eq. (A.16)

$$I_{(S^7/\mathbb{Z}_m)/\mathbb{Z}_2:sp}^{(0)}(x) = \frac{x(3 + 2x^{1/2} + 2x - 2x^{5/2} - x^3)}{(1 - x^2)^2}, \quad (4.28)$$

after a careful analysis on the suitable \mathbb{Z}_2 modding of the gravity index on $AdS_4 \times (S^7/\mathbb{Z}_m)/\mathbb{Z}_2$. As happened in $U(N)$ case, twisted sector comes only from zero monopole charge sector. The twisted index is given by

$$\begin{aligned} I_{O(\infty):m:twisted:sp}^{(0)}(x) &= I_{O(\infty):m:sp}^{(0)}(x) - I_{(S^7/\mathbb{Z}_m)/\mathbb{Z}_2:sp}^{(0)}(x), \\ &= (2m^2 + m)A(x) + (2m^2 - m - 1)B(x). \end{aligned} \quad (4.29)$$

Where $A(x), B(x)$ are given by

$$A(x) = \frac{x}{(1 - x)^2}, \quad B(x) = \frac{2x^{3/2}}{(1 - x)^2(1 + x)}. \quad (4.30)$$

They are provided in [17] by analyzing 7-d SYM on $AdS_4 \times S^3$. The index for the 7d $U(1)$ SYM is

$$I_{7d\ U(1)\ \text{SYM}}(x) = A(x) + B(x). \quad (4.31)$$

Note that $A(x)$ denotes the spectrum with the integer value while $B(x)$ represents spectrum with half-integer value.¹ The above expression of the twisted sector implies that we keep the adjoint of the gauge group $Sp(2m)$ on D6 branes for the integer spectrum while keeping antisymmetric representation for half-integer spectrum. Note that the antisymmetric representation of $Sp(2m)$ is made of a irreducible representation of $\dim\ m(2m-1)-1$ and one singlet. The lowest integer spectrum represents the the gauge degrees of freedom on AdS_4 so this must be the adjoint representation of $\dim\ m(2m+1)$ for $Sp(2m)$, which is the gauge group of D6s in the brane setup. The factor $(2m^2 \pm m)$ in twisted sector index can also be understood by explicitly constructing local operators in $\mathcal{N} = 4\ O(2N)$ SYM theory. Let $(A_I, B_I)|_{I=1,\dots,m} \in (2N, 2\bar{N})$ be m fundamental hypermultiplets and (Φ_1, Φ_2) be an adjoint hyper-multiplet multiplet in the SYM. Since $2N = 2\bar{N}$, combining A_I and B_I we define $2m$ fundamental chiral multiplets $F_I|_{I=1,\dots,2m}$ as $(I = 1, \dots, 2m)$

$$F_I = A_I, \quad F_{2m+I} = B_I. \quad (4.32)$$

Then lower power in twisted sector index come from

$$\begin{aligned} F_I^T F_J &: (2m^2 + m)x, \\ F_I^T \Phi_1 F_J, \ F_I^T \Phi_2 F_J &: 2(2m^2 - m)x^{3/2}. \end{aligned} \quad (4.33)$$

Note that $F_I^T \Phi_{1,2} F_J = -F_J^T \Phi_{1,2} F_I$ due to the property $\Phi_{1,2}^T = -\Phi_{1,2}$. F-term equation for Φ (adjoint chiral multiplet in $\mathcal{N} = 4$ vector-multiplet) will kill one combination of local operators of the form $F_I^T \Phi F_J$ and give correct factor $(2m^2 - m - 1)$.

One can also compare the non-zero monopole charge part of field theory large N index, $I_{O(2N):m}(x)$, and its corresponding SUGRA index, $I_{(S^7/\mathbb{Z}_m)/\mathbb{Z}_2}(x)$, and find exact match using the similar trick used in $U(2N)$ case.

Now let us turn our attention to the $Sp(2N) \times SO(2m)$ case, the similar analysis gives the twisted sector as, see eq. (A.16) and (B.20)

$$\begin{aligned} I_{Sp(\infty):m:\text{twisted}:sp}(x) &= I_{Sp(\infty):m:sp}^{(0)}(x) - I_{(S^7/\mathbb{Z}_m)/\mathbb{Z}_2:sp}^{(0)}(x) \\ &= (2m^2 - m)A(x) + (2m^2 + m - 1)B(x). \end{aligned} \quad (4.34)$$

Again this has the simple interpretation in terms of the gauge group $SO(2m)$ for D6 branes. The integer spectrum has the adjoint representation of $\dim\ m(2m-1)$ for

¹ $A(x)$ is the index from states in the 7d SYM theory with $\beta = 1$ and $B(x)$ is the index from states with $\beta = -1$. β is the generator of \mathbb{Z}_2 in $AdS_4 \times S^3/\mathbb{Z}_2$, which has the singular locus of the gravity background.

$SO(2m)$, while the half integer spectrum has the traceless symmetric representation of $SO(2m)$. One can also find operators in the $\mathcal{N} = 4$ $Sp(2N)$ SYM corresponding to twisted sector index.

$$\begin{aligned} F_{[I}^T \mathbb{J} F_{J]} &: (2m^2 - m)x, \\ F_{(I}^T \mathbb{J} \Phi_1 F_{J)}, F_{(I}^T \mathbb{J} \Phi_2 F_{J)} &: 2(2m^2 + m)x^{3/2}. \end{aligned} \quad (4.35)$$

\mathbb{J} denote the skew-symmetric form of Sp -group. Note that $F_I^T \mathbb{J} F_J = -F_J^T \mathbb{J} F_I$ due to the property $\mathbb{J}^T = -\mathbb{J}$ and $F_I^T \mathbb{J} \Phi_{1,2} F_J = F_J^T \mathbb{J} \Phi_{1,2} F_I$ due to the property $\mathbb{J} \Phi_{1,2} + \Phi_{1,2}^T \mathbb{J} = 0$. All these are consistent with the orientifold projection $SU(2m) \rightarrow Sp(2m)/SO(2m)$.

Finally for the $SO(2N+1) \times Sp(2m)$ case, the twisted sector contribution is given by eq. (4.29) as it should be since twisted sectors are coming from D6 strings, which has the same $Sp(2m)$ projection for both $O(2N)$ and $SO(2N+1)$ theories.

The index computation strongly suggests that in the large N limit, the IR superconformal theories of $\mathcal{N} = 8$ $O(2N)$, $Sp(2N)$, $SO(2N+1)$ super Yang-Mills theory are dual to the gravity theory on $AdS_4 \times S^7/\mathbb{Z}_2$. As the ABJ(M) models of the $U(N)_2 \times U(N)_{-2}$ and $U(N)_2 \times U(N+1)_{-2}$ groups also have the same gravitational dual. In order to tell which theory is mapped to which one, we need the information on finite N . Such information is given by \mathbb{Z}_2 orbifolded theory. In the Type IIB Hanany-Witten setup with D3/NS5/ (1,2) 5brane, introducing m NS 5branes gives rise to $\mathbb{C}^2/\mathbb{Z}_m$ orbifold singularities. If we consider 2 NS5 branes, this also leads to $U(N)_0$ factor in addition to \mathbb{Z}_2 orbifold. We already saw that $O(2N)$ SYM with one fundamental hyper is mapped to $U(N)_2 \times U(N)_{-2} \times U(N)_0$ SCSM while $USp(2N)$ SYM is mapped to $U(N)_2 \times U(N+1)_{-2} \times U(N)_0$ SCSM. Since we can take the large N limit for these theories, this tells us that for \mathbb{Z}_2 invariant sectors we have to identify $O(2N)$ theory with $U(N)_2 \times U(N)_{-2}$ and $USp(2N)$ theory with $U(N+1)_2 \times U(N)_{-2}$. Hence this identification is natural for the whole theory. Thus our claim is that $\mathcal{N} = 8$ $O(2N)$ SYM flows to $U(N)_2 \times U(N)_{-2}$ ABJM and $\mathcal{N} = 8$ $SO(2N+1)$, $Sp(2N)$ SYM flows to $U(N+1)_2 \times U(N)_{-2}$ ABJ theory. In the case of $Sp(2N)$ we are assuming $\vartheta = 0$ in (2.1).

Now consider the Higgsing pattern of ABJ(M) theory. Starting from $U(N)_2 \times U(N)_{-2}$ ABJM this can be Higgsed down to $U(1)_2 \times U(1)_{-2} \times U(1)^{N-1}$ where the last factor represents the product of the $U(1)$ Yang-Mills theory. This can be achieved by giving the scalar vev of ABJM theory

$$Z_4 = \bar{Z}^4 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & R_1 & 0 & \cdots \\ 0 & 0 & R_2 & \cdots \end{pmatrix}. \quad (4.36)$$

We already saw that $U(1)_2 \times U(1)_{-2}$ ABJM theory is the IR limit of $O(2)$ or $Sp(2)$ with $\vartheta = \pi$. And the Higgsing pattern is $O(2N) \rightarrow O(2) \times U(1)^{N-1}$ and $Sp(2N) \rightarrow$

$Sp(2) \times U(1)^{N-1}$. We conclude that the IR limit of $\mathcal{N} = 8$ $O(2N), Sp(2N)$ with $\vartheta = \pi$ is given by $U(N)_2 \times U(N)_{-2}$ ABJM theory. For $U(N)_2 \times U(N+1)_{-2}$ ABJ theory, it is Higgsed to $U(1)_2 \times U(2)_{-2} \times U(1)^{N-1}$ where the last factor represents the product of the $U(1)$ Yang-Mills theory. Since the IR limit of $SO(3)$ and $Sp(2)$ with $\vartheta = 0$ is given by $U(1)_2 \times U(2)_{-2}$ ABJ theory, the IR limit of $SO(2N+1)$ and $Sp(2N)$ with $\theta = 0$ should be given by $U(N)_2 \times U(N+1)_{-2}$ ABJ theory.

5. Mirror symmetry and $\mathcal{N} = 4$ Supersymmetric Chern-Simons theory

5.1 Basic setup and the index computation

In three-dimensions, there are important classes of $\mathcal{N} = 4$ superconformal field theories which are extensively discussed in the context of mirror symmetry. These can be described in Hanany-Witten setup as a collection of D3/NS5/D5 branes. The mirror symmetry is realized as the S-dual transformation of $SL(2, Z)$ of Type IIB theory, which interchanges NS5 and D5². In the field theory side, this interchanges Coulomb and Higgs branch. At the origin of the moduli space where the Coulomb and the Higgs meet, we have 3d superconformal field theory. The natural question is if these superconformal field theories admit Chern-Simons type description [23, 24]. The answer is positive and the subsequent computation gives an impressive confirmation.

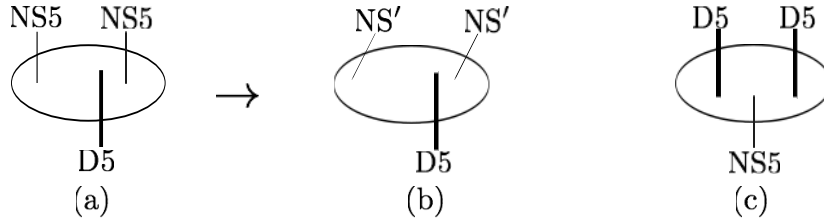


Figure 1: (a) A configuration of N D3s, 2 NS5s, and one D5. (b) T-dual transformation of (a). (c) S-dual transformation of (a).

Let us start with the simplest example, N D3 branes with 2 NS5s and one D5 as shown in Fig 1 (a). This has $\mathcal{N} = 4$ supersymmetry in 3-d and we have $U(N) \times U(N)$ YM with hypermultiplets transforming $(N, \bar{N}), (\bar{N}, N), (1, N)$. If we take the T-dual transformation $\tau \rightarrow \tau + 1$ of $SL(2, Z)$ in Type IIB setting, D5 branes are invariant while NS5 branes are turned into (NS5, D5)=(1,1) brane, which we will call NS' brane subsequently. These are the configurations considered by Imamura and Kimura [27]. We have $\mathcal{N} = 4$ SCSM with the gauge group $U(N)_1 \times U(N)_0 \times U(N)_{-1}$ with bi-fundamental hypers in

$$(N, \bar{N}, 1), (N, \bar{N}, 1), (\bar{N}, 1, N). \quad (5.1)$$

²For later purpose, we had better fix the world volume directions of D3/NS5/D5 branes. D3 spans (0123), D5 spans (012456) and NS5 spans (012789)

The subscript in the gauge group denotes Chern-Simons level. This is shown in Fig 1 (b). This suggests that the $\mathcal{N} = 4$ SYM of Fig 1 (a) flows to the SCSM of Fig 1 (b). Alternatively, one can take S-dual transformation from Fig 1 (a) to obtain Fig 1 (c), exchanging NS5 and D5 branes. The resultant theory is $U(N)$ YM with two fundamental and one adjoint hyper-multiplets. The index computation confirms this.³ For example, one obtains

$$I_{YM \oplus 1 \text{ fund. hyper}}^{U(1) \times U(1)} = I_{YM \oplus 2 \text{ fund. hypers}}^{U(1)} = I_{SCS}^{U(1)_1 \times U(1)_0 \times U(1)_{-1}} \\ = 1 + 2x^{1/2} + 9x + 14x^{3/2} + 22x^2 + 20x^{5/2} + 25x^3 + 34x^{7/2} + 62x^4 + 74x^{9/2} + O(x^5).$$

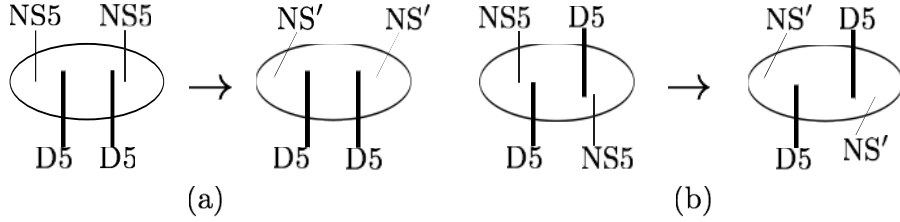


Figure 2: Two possible configurations of 2 NS5s and 2 D5s

Now consider another example where now we have 2 NS5s and 2 D5s. Note that these are self-mirror configurations. As shown in Fig 2 (a) (b) there are two possible configurations of NS/D5. One is NS-NS-D5-D5 as appearing in Fig 2 (a) and the other is NS-D5-NS-D5 as appearing in Fig 2 (b). In (a), the first one gives $U(N) \times U(N)$ Yang-Mills with hyper-multiplets in $(N, \bar{N}), (\bar{N}, N), 2(1, N)$. Under the T-dual transformation it turns into Chern-Simons quiver theory $U(N)_1 \times U(N)_0 \times U(N)_{-1} \times U(N)_0$ with bi-fundamental hyper-multiplets in

$$(N, \bar{N}, 1, 1), (1, N, \bar{N}, 1), (1, 1, N, \bar{N}), (\bar{N}, 1, 1, N). \quad (5.2)$$

The index computation gives the same result. As an example we exhibit the index with $N = 1$

$$I_{CS}^{U(1)_1 \times U(1)_0 \times U(1)_{-1} \times U(1)_0} = I_{YM}^{U(1) \times U(1)} = 1 + 12x + 42x^2 + 48x^3 + 115x^4 + 188x^5 + O(x^6). \quad (5.3)$$

In Fig 2 (b), the first one gives $U(N) \times U(N)$ Yang-Mills theory with hyper-multiplets in $(N, \bar{N}), (\bar{N}, N), (N, 1), (1, N)$. Under the T-dual transformation, it is mapped to $U(N)_1 \times U(N)_{-1} \times U(N)_1 \times U(N)_{-1}$ quiver Chern-Simons theory with bi-fundamental hyper-multiplets in (5.2). Again for $N = 1$, the index computation

³For $\mathcal{N} = 4$ abelian theories related by mirror symmetry, one has analytic proof for the equality of the index at [28]. It would be interesting to find the similar proof for the cases handled in this paper.

gives

$$\begin{aligned}
I_{CS}^{U(1)_1 \times U(1)_{-1} \times U(1)_1 \times U(1)_{-1}} &= I_{YM'}^{U(1) \times U(1)} \\
&= 1 + 8x + 8x^{3/2} + 18x^2 + 16x^{5/2} + 28x^3 + 63x^4 + 80x^{9/2} + 56x^5 + O(x^{11/2})
\end{aligned} \tag{5.4}$$

where now the hyper-multiplets in the YM' have the charges $(1, -1) \oplus (-1, 1) \oplus (1, 0) \oplus (0, 1)$ under the gauge group, in contrast to that the hyper-multiplets in YM in (5.3) have charges $(1, -1) \oplus (-1, 1) \oplus 2(1, 0)$.

The interesting feature is that they have the same moduli space yet different index. One can easily see why this is so. When one tries to exchange NS and D5 (or NS' and D5) to obtain one brane configuration from the other, we create D3 brane between NS' and D5 [10]. This suggests an equivalence among $\mathcal{N} = 4$ Chern-Simons quiver theories

$$U(N)_1 \times U(N)_0 \times U(N)_{-1} \times U(N)_0 \sim U(N+1)_1 \times U(N)_{-1} \times U(N)_1 \times U(N)_{-1} \tag{5.5}$$

as appearing in Fig 3. Indeed the index computation confirms this, for example

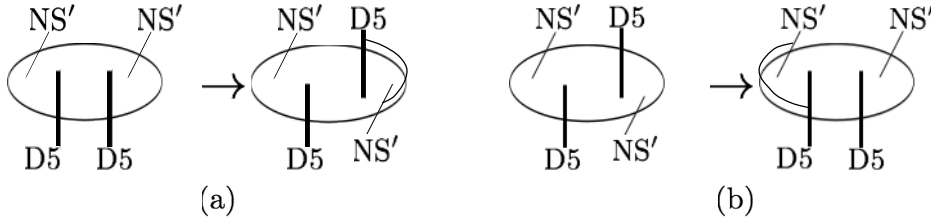


Figure 3: D3-brane creation due to Hanany-Witten effect

$N = 1$ case gives,

$$\begin{aligned}
I_{CS}^{U(1)_1 \times U(1)_{-1} \times U(1)_1 \times U(1)_{-1}}(x) &= I_{CS}^{U(2)_1 \times U(1)_0 \times U(1)_{-1} \times U(1)_0}(x) \\
I_{CS}^{U(1)_1 \times U(1)_0 \times U(1)_{-1} \times U(1)_0}(x) &= I_{CS}^{U(2)_{-1} \times U(1)_1 \times U(1)_{-1} \times U(1)_1}(x)
\end{aligned}$$

where the explicit forms are given in eq. (5.3) and (5.4).

For Chern-Simons theories with higher level, similar brane creation effect can occur. Taking account of this, one can see the equivalence of various theories, which can be regarded as the $\mathcal{N} = 4$ generalization of the $\mathcal{N} = 6$ dualities in [6]. For example, the following two $\mathcal{N} = 4$ SCSMs are expected to be equivalent,

$$U(N)_k \times U(N)_{-k} \times U(N)_k \times U(N)_{-k} \sim U(N + |k|)_k \times U(N)_0 \times U(N)_{-k} \times U(N)_0 .$$

An explicit example for $N = 1, k = 2$ is

$$I_{CS}^{U(1)_2 \times U(1)_{-2} \times U(1)_2 \times U(1)_{-2}} = I_{CS}^{U(3)_2 \times U(1)_0 \times U(1)_{-2} \times U(1)_0} = 1 + 4x + 18x^2 + 16x^3 + 35x^4 + O(x^5).$$

Now the pattern for general configurations is obvious. In the appendix we look for the partition function of these theories and found agreements. We also turn on FI and mass parameters and show how these are mapped under the dualities between SYM and SCSM and provide subsequent interpretation in the next subsection.

5.2 FI and mass parameters

In this subsection, we generalize the result by adding mass and FI parameters. The following discussion includes the ABJM and $\mathcal{N} = 8$ SYM considered in [5] as a special case $m = n = 1$, where m (n) is the number of D5's (NS5 or NS's). Since the FI and mass parameter do not carry the color index, we will consider the map of parameters of abelian theories.

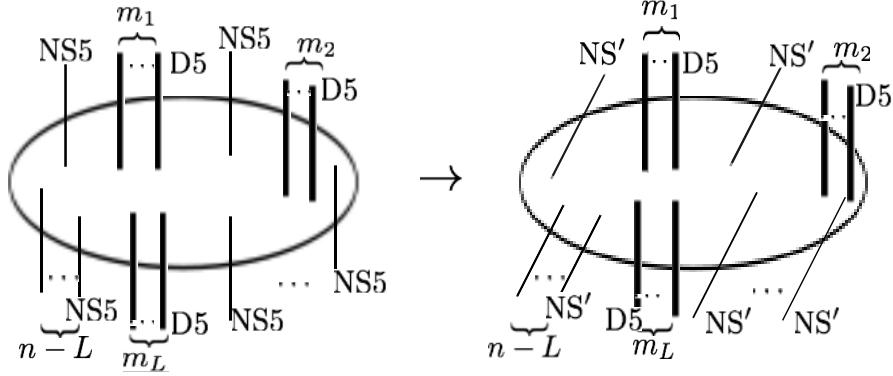


Figure 4: (a) A generic Hanany-Witten set up. (b) T-dual transformation from (a)

As the first example, we consider the case that one NS5 and one D5 are alternating, i.e. $m = n = L$, $m_1 = \dots = m_n = 1$, in the Fig. 4. The parameters of interest are

YM	$U(1)^n$	bi-fundamental masses ω_I	$(I = 1, \dots, n)$	(5.6)
		fundamental masses μ_I		
		FI parameters η_I		
CS	$U(1)^{2n}$	bi-fundamental masses ξ_I	$(I = 1, \dots, 2n)$	(5.7)
		FI parameters ζ_I		

The Chern-Simons level is alternating 1 and -1 , $U(1)_1 \times U(1)_{-1} \times \dots \times U(1)_{-1}$.

Some parameters in (5.6) are not independent [5]. One can see it from the partition function. If we turn on all parameters, the partition functions are changed to

$$Z_{YM}(\omega, \mu, \eta) = \int (d\sigma)^n \frac{e^{2\pi i \sum_{I=1}^n \eta^I \sigma^I}}{\prod_{I=1}^n \cosh(\pi(\sigma^I - \sigma^{I+1} + \omega_I)) \prod_{I=1}^n \cosh(\pi(\sigma^I + \mu^I))},$$

$$Z_{CS}(\xi, \zeta) = \int (d\sigma)^{2n} \frac{e^{\pi i((\sigma^1)^2 - (\sigma^2)^2 + \dots + (\sigma^{2n-1})^2 - (\sigma^{2n})^2) + 2\pi i \sum_{I=1}^{2n} \zeta^I \sigma^I}}{\prod_{I=1}^{2n} \cosh(\pi(\sigma^I - \sigma^{I+1} + \xi_I))}.$$

However, by the constant shift of integral variables, one can absorb one of the fundamental masses of YM. In the same way, one can set the bi-fundamental masses of YM and CS to be same respectively,

$$\omega_1 = \omega_2 = \dots = \omega_n := \omega, \quad \xi_1 = \xi_2 = \dots = \xi_{2n} := \xi. \quad (5.7)$$

The FI parameters of CS can be set to satisfy

$$\sum_{I=1}^n \zeta^{2I-1} = \sum_{I=1}^n \zeta^{2I} := n\zeta \quad (5.8)$$

for some fixed ζ , which becomes obvious in (5.11) after changes of variables. Thus the number of independent parameters are $2n$; 1 bi-fundamental mass, $n-1$ fundamental masses, and n FI parameters in YM; 1 bi-fundamental mass and $2n-1$ FI parameters in CS.

Imposing (5.7), the partition functions become

$$Z_{YM}(\omega, \eta) = \int (d\sigma)^n \frac{e^{2\pi i \sum_{I=1}^n \eta^I \sigma^I}}{\prod_{I=1}^n \cosh(\pi(\sigma^I - \sigma^{I+1} + \omega)) \prod_{I=1}^n \cosh(\pi(\sigma^I + \mu^I))}, \quad (5.9)$$

$$Z_{CS}(\xi, \zeta) = \int (d\sigma)^{2n} \frac{e^{\pi i((\sigma^1)^2 - (\sigma^2)^2 + \dots + (\sigma^{2n-1})^2 - (\sigma^{2n})^2) + 2\pi i \sum_{I=1}^{2n} \zeta^I \sigma^I}}{\prod_{I=1}^{2n} \cosh(\pi(\sigma^I - \sigma^{I+1} + \xi))}. \quad (5.10)$$

One can use formulas in section C.1 to rewrite $Z_{CS}(\xi, \zeta)$ as

$$\begin{aligned} Z_{CS}(\xi, \zeta) &= \int (d\sigma)^{2n} (d\tau)^{2n} \frac{e^{\pi i((\sigma^1)^2 - (\sigma^2)^2 + \dots + (\sigma^{2n-1})^2 - (\sigma^{2n})^2) + 2\pi i \sum_{I=1}^{2n} (\tau^I \xi + \sigma^I (\tau^I - \tau^{I-1} + \zeta^I))}}{\prod_{I=1}^{2n} \cosh(\pi \tau^I)} \\ &= \int (d\tau)^{2n} \frac{e^{2\pi i(\sum_{a=1}^n (\tau^{2a+1} - \tau^{2a-1}) \tau^{2a} + 2\pi i \sum_{a=1}^n (\tau^{2a} (\zeta^{2a} + \zeta^{2a+1} + \xi) - \tau^{2a-1} (\zeta^{2a-1} + \zeta^{2a} - \xi))}}{\prod_{a=1}^n \cosh(\pi \tau^{2a}) \prod_{a=1}^n \cosh(\pi \tau^{2a-1})} \\ &= \int (d\kappa)^n \frac{e^{2\pi i \sum_{I=1}^n \kappa^I (\zeta^{2I-1} + \zeta^{2I} - \xi)}}{\prod_{I=1}^n \cosh(\pi \kappa^I) \cosh(\pi(\kappa^I - \kappa^{I+1} + \zeta^{2I} + \zeta^{2I+1} + \xi))} \\ &= \int (d\tau)^n \frac{e^{2\pi i \sum_{I=1}^n \tau^I (\zeta^{2I-1} + \zeta^{2I} - \xi)}}{\prod_{I=1}^n \cosh(\pi(\tau^I - \tau^{I+1} + \mu^I - \mu^{I+1} + \zeta^{2I} + \zeta^{2I+1} + \xi)) \cosh(\pi(\tau^I + \mu^I))}. \end{aligned} \quad (5.11)$$

In the last two lines, we define $\kappa^I = -\tau^{2I-1}$ then recycle τ for $\tau^I := \kappa^I + \mu^I$. The equalities hold up to an overall phase. Comparison of (5.9) and (5.12) gives the following map between mass and FI parameters

$$\begin{aligned} \omega &= (2\zeta + \xi) \\ \mu^I - \mu^{I+1} &= 2\zeta - (\zeta^{2I} + \zeta^{2I+1}), \quad (I = 1, \dots, n) \\ \eta^I &= (\zeta^{2I-1} + \zeta^{2I} - \xi), \quad (I = 1, \dots, n). \end{aligned} \quad (5.13)$$

For $(n, m) = (1, 1)$, it reduces to $\mathcal{N} = 8$ SYM/ ABJM result in [5],

$$\omega = 2\zeta + \xi, \quad \eta = 2\zeta - \xi. \quad (5.14)$$

The peculiar feature is that in the mapping from YM type to SCSM theory, FI term and mass term are mixed up with each other. In the Yang-Mills theory,

there's $SO(4) = SU(2)_L \times SU(2)_R$ R symmetry denoted by R_{YM} . Under this, FI term transforms as (1,3) and mass term transforms as (3,1). In the Hanany-Witten setup, fundamental mass term is given by the transverse location x_4, x_5, x_6 of D5 branes while FI term is the transverse location x_7, x_8, x_9 of NS5 branes. Now in the brane setup it's obvious that among $SU(2)_L \times SU(2)_R$ R_{YM} symmetry only its diagonal combination survives since the NS' brane is rotated in the 456-789 planes with respect to NS brane. Hence it's natural that FI term and mass term in the YM theory is mixed up in the SCSM setting. The $SO(4)'_R = SU(2)'_L \times SU(2)'_R$ symmetry in the SCSM theory of interest arise where the R-symmetry transformation on the hypermultiplets and the twisted hypermultiplets are different, which cannot be seen in SYM theory. Note that we have $2n - 1$ FI parameters which should transform as (3,1) or (1,3) under $SO(4)_{R'} = SU(2)_{L'} \times SU(2)_{R'}$. The distinction is not important since the interchange of $SU(2)_{L'}$ and $SU(2)_{R'}$ interchanges hypermultiplets and twisted hypermultiplets, which leads to the same theory.

As the second example, we consider the case that all D5 branes are between the first and the second NS5-branes. In Fig. 4, it corresponds to $L = 1, m_1 = m$. The parameters of interest are

YM	$U(1)^n$	bi-fundamental mass ω_I	$(I = 1, \dots, n)$
		fundamental mass μ_a	$(a = 1, \dots, m)$
		FI parameters η_I	$(I = 1, \dots, n)$
CS	$U(1)^{n+m}$	bi-fundamental mass ξ_I	$(I = 1, 2, \dots, n + m)$
		FI parameters ζ_I	

The Chern-Simons level is $\vec{k} = (1, 0, \dots, 0, -1, 0, \dots, 0)$, nonvanishing for the first and the $(m+1)$ th gauge group. Again, not all parameters are independent. The constant shift of integral variables in the partition function can impose

$$\begin{aligned} \omega_1 = \omega_2 = \dots = \omega_n &:= \omega, & \mu_1 &= 0, \\ \xi_1 = \xi_{m+1} &:= \xi, & \xi_I &= 0 \quad \text{for } I \neq 1, m+1, & \zeta_1 = \zeta_{m+1} &:= \zeta \end{aligned}$$

Thus the number of independent parameters is $(n + m)$; 1 bi-fundamental mass, $m - 1$ fundamental mass, and n FI parameters in YM; 1 bi-fundamental mass and $n + m - 1$ FI parameters in CS.

We now repeat the derivation. The Chern-Simons partition function can be written as an integral over $(n + m)$ variables,

$$Z_{CS}(\xi, \zeta) = \int (d\sigma)^{n+m} \frac{e^{\pi i((\sigma^1)^2 - (\sigma^{m+1})^2) + 2\pi i \sum_{I=1}^{n+m} \zeta^I \sigma^I}}{\prod_{I=1}^{n+m} \cosh(\pi(\sigma^I - \sigma^{I+1} + \xi))}.$$

We use (C.6) to introduce $(n + m)$ τ -variables to Chern-Simons partition function, then integrate out σ and τ variables in turn except $\sigma^1, \sigma^{m+1}, \tau^1, \tau^{m+1}$. The Gaussian

integration of σ^1 and σ^{m+1} results in

$$Z_{CS}(\xi, \zeta_I) = \int (d\tau^1 d\tau^{m+1}) \frac{1}{\cosh(\pi\tau^1) \cosh(\pi\tau^{m+1})} \frac{e^{2\pi i((-\xi + (\zeta_1 + \zeta_{m+1}))\tau^1 + (\xi + (\zeta_1 + \zeta_{m+1}))\tau^{m+1})}}{(\prod_{I=2}^m \cosh \pi(\tau^1 + \sum_{\alpha=2}^I \zeta_\alpha)) (\prod_{a=2}^n \cosh \pi(\tau^{m+1} - \sum_{\alpha=m+2}^{m+a} \zeta_\alpha))}. \quad (5.15)$$

On the other hand, the YM partition function is given by

$$Z_{YM}(\vec{\eta}, \omega, \vec{\mu}) = \int (d\sigma)^n \frac{e^{2\pi i \sum_{I=1}^n \eta_I \sigma^I}}{\prod_I \cosh(\pi(\sigma^I - \sigma^{I+1} + \omega)) \cosh(\pi\sigma^1) \prod_{a=2}^m \cosh(\pi(\sigma^1 + \mu_a))}.$$

We use (C.6) to introduce n τ -variables, then integrate out σ and τ variables in turn except σ^1, τ^1 . It results in

$$Z_{YM}(\vec{\eta}, \omega, \vec{\mu}) = \int d\sigma d\tau \frac{e^{2\pi i(\sum_{I=1}^n \eta_I \sigma + 2\pi i(n\omega)\tau)}}{\prod_{I=2}^n \cosh(\pi(\tau - \sum_{\alpha=2}^I \eta_\alpha)) \cosh(\pi\tau) \prod_{a=1}^m \cosh(\pi(\sigma + \mu_a))}. \quad (5.16)$$

Then comparison of (5.15) and (5.16) gives the following map

$$\begin{aligned} n\omega &= 2\zeta + \xi, & \sum_{I=1}^n \eta_I &= 2\zeta - \xi, \\ \mu_a &= \zeta_a, & (a = 2, 3, \dots, m), & \quad \eta_I = \zeta_{m+I}, \quad (I = 2, 3, \dots, n) \end{aligned} \quad (5.17)$$

Again, for $(n, m) = (1, 1)$, it reduces to the result of $\mathcal{N} = 8$ SYM/ ABJM in (5.14). Again we can observe the mixing between FI and mass terms.

6. Concluding Remarks

We found the equivalence between the low energy superconformal field theory of $\mathcal{N} = 8$ super Yang-Mills theories of classical groups ABCD and the superconformal ABJ(M) models of the Chern-Simons level $k = 1, 2$. The supportive evidence is found from the match of superconformal indices.

We also partially find the match in the partition function as shown in Appendix C. However, we have not exhausted the calculation of the indices for the $\mathcal{N} = 8$ superconformal field theories. Besides the IR limit of the SYM theories of $SU(N)$, $SO(2N)$ and exceptional gauge groups, there are also the Chern-Simons matter model of $SU(N)_k \times SU(N)_{-k}$ with $k = 1, 2$ and BLG model of arbitrary k [25, 26], which has been studied extensively. There may be more $\mathcal{N} = 8$ superconformal field theories besides what has been discussed. It remains to be seen.

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A. Index over gravitons in $AdS_4 \times S^7/\Gamma$

In this section, we summarize relevant results on graviton index on $AdS_4 \times S^7/\Gamma$ with discrete quotient Γ . The index of single graviton in $AdS_4 \times S^7$ is given by [7]

$$I_{S^7:sp}(x, y_1, y_2, y_3) = \text{Tr}_{(\text{single graviton})}(-1)^F x^{\epsilon_0+j_3} y_1^{h_1} y_2^{h_2} y_3^{h_3} = \frac{(\text{numerator})}{(\text{denominator})}, \quad (\text{A.1})$$

where

$$\begin{aligned} (\text{numerator}) &= \sqrt{y_1 y_2 y_3} (1 + y_1 y_2 + y_2 y_3 + y_3 y_1) x^{\frac{1}{2}} - \sqrt{y_1 y_2 y_3} (y_1 + y_2 + y_3 + y_1 y_2 y_3) x^{\frac{7}{2}} \\ &\quad + (y_1 y_2 + y_2 y_3 + y_3 y_1 + y_1 y_2 y_3 (y_1 + y_2 + y_3)) (x^3 - x), \\ (\text{denominator}) &= (1 - x^2) (\sqrt{y_3} - \sqrt{x y_1 y_2}) (\sqrt{y_1} - \sqrt{x y_2 y_3}) (\sqrt{y_2} - \sqrt{x y_3 y_1}) (\sqrt{y_1 y_2 y_3} - \sqrt{x}). \end{aligned} \quad (\text{A.2})$$

$\{h_i\}_{i=1,\dots,4}$ denote four Cartans of $SO(8)$ isometry in S^7 . ϵ_0 and j_3 are two Cartans of $SO(2, 3)$ isometry in AdS_4 which are called energy and spin respectively. The $SO(8)$ generators act on $\mathbb{C}^4 = \text{Cone}(S^7)$ as follows

$$\begin{aligned} h_1 &= \text{diag}\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \quad h_2 = \text{diag}\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \\ h_3 &= \text{diag}\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad h_4 = \text{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right). \end{aligned} \quad (\text{A.3})$$

Only gravitons satisfying following BPS bound contribute to the index.

$$\epsilon_0 = j_3 + h_4. \quad (\text{A.4})$$

The graviton index on $AdS_4 \times S^7/\Gamma$ can be obtained by keeping only the contribution from Γ -invariant gravitons.

Consider the case when $\Gamma = \mathbb{Z}_m$, whose generator $\exp(\frac{2\pi i}{m}h)$ with $h = h_2 - h_1$ act on the \mathbb{C}^4 as

$$h = \text{diag}(0, 0, 1, -1). \quad (\text{A.5})$$

To keep contribution from \mathbb{Z}_m -invariant gravitons, it's convenient to introduce to chemical potential y for the charge h .

$$\begin{aligned} I_{S^7:sp}(x, y) &= I_{S^7:sp}(x, y_1 = 1/y, y_2 = y, y_3 = 1), \\ &= \sum_{n \in \mathbb{Z}} I_{S^7:sp}^{(n)}(x) y^n, \end{aligned} \quad (\text{A.6})$$

where

$$I_{S^7:sp}^{(n)}(x) = \begin{cases} \frac{2(x^{1/2} + x - x^{5/2})}{1 - x^{1/2} - x^2 + x^{5/2}}, & n = 0 \\ x^{|n|/2} \frac{(1 + x^{1/2} + x)^2}{1 - x^2}, & n \neq 0. \end{cases} \quad (\text{A.7})$$

Then, the graviton index in $AdS_4 \times S^7/\mathbb{Z}_m$ can be written as

$$I_{S^7/\mathbb{Z}_m;sp}(x, y) = I_{S^7;sp}^{(0)} + \sum_{n \neq 0} y^{mn} I_{S^7;sp}^{(mn)}(x). \quad (\text{A.8})$$

Consider the case when $\Gamma = \langle \alpha, \beta \rangle$, discrete group generated by α, β defined in eq. (4.20). In this case, we introduce two chemical potentials z_1 and z_2 for charges J_3 and J'_3 , for convenience. J_i, J'_i are generators of two $SU(2)$ s acting on two \mathbb{C}^2 factors in \mathbb{C}^4 respectively. That is

$$\begin{aligned} J_3 &= \frac{1}{2}(h_1 + h_2) = \text{diag}\left(\frac{1}{2}, -\frac{1}{2}, 0, 0\right), \\ J'_3 &= \frac{1}{2}(h_2 - h_1) = \text{diag}\left(0, 0, \frac{1}{2}, -\frac{1}{2}\right). \end{aligned} \quad (\text{A.9})$$

We use the normalization for $SU(2)$ generators such that J_3 has eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$ in minimal (fundamental) representation. The graviton index in $AdS_4 \times S^7$ can be written in terms of two chemical potential z_1, z_2 as follow

$$\begin{aligned} I_{S^7;sp}(x, z_1, z_2) &= I_{S^7;sp}(x, y_1 = \sqrt{z_1 z_2}, y_2 = \sqrt{z_1/z_2}, y_3 = 1), \\ &= \sum I_{S^7}^{(J, J')}(x) \chi_J^{SU(2)}(z_1) \chi_{J'}^{SU(2)}(z_2). \end{aligned} \quad (\text{A.10})$$

Using the $SU(2)^2$ isometry in S^7 geometry, the index can be expanded in terms of two $SU(2)$ characters, $\chi_J(z_1)$ and $\chi_{J'}(z_2)$. Recall that the discrete group Γ is generated by two elements $\alpha = \exp(\frac{4\pi i J_3}{2m}) \otimes \mathbb{I}$ and $\beta = \exp(\pi i J_2) \otimes \exp(2\pi i J'_3)$. Γ -invariant states can be divided into following two types (depending on $J_3 = 0$ or $J_3 \neq 0$),

$$\begin{aligned} \text{A-type : } & |J, J_3 = 0\rangle \otimes |J', J'_3\rangle \text{ with } (-1)^{J+2J'} = 1, \\ \text{B-type : } & (|J, J_3 \neq 0\rangle + |J, -J_3\rangle) \otimes |J', J'_3\rangle \text{ with } J_3 \in m\mathbb{Z} \text{ and } J'_3 \in \mathbb{Z}, \text{ or} \\ & (|J, J_3 \neq 0\rangle - |J, -J_3\rangle) \otimes |J', J'_3\rangle \text{ with } J_3 \in m\mathbb{Z} \text{ and } J'_3 \in \mathbb{Z} + \frac{1}{2}. \end{aligned} \quad (\text{A.11})$$

Here, states are represented by their total angular momentum J and z -component J_z of two $SU(2)$. Other quantum numbers are irrelevant and thus suppressed. It's easy to see that these states are invariant under α . β -invariance of states in A-type can be shown as follows,

$$\begin{aligned} \beta \cdot (|J, J_3 = 0\rangle \otimes |J', J'_3\rangle) &= (-1)^J (-1)^{2J'_3} (|J, J_3 = 0\rangle \otimes |J', J'_3\rangle), \\ &= (-1)^{J+2J'} (|J, J_3 = 0\rangle \otimes |J', J'_3\rangle). \end{aligned} \quad (\text{A.12})$$

Here we use the fact that $(-1)^{\pi i J_2} |J_3 = 0\rangle = (-1)^J |J_3 = 0\rangle$ for $J \in \mathbb{Z}$ and $(-1)^{J'_3} = (-1)^{J'}$. To see the β -invariance of the states in B-type, one need to note that

$$\{\exp(\pi i J_2), J_3\} = 0, \quad \text{when } J \in \mathbb{Z}. \quad (\text{A.13})$$

Thus β flips the sign of quantum number J_3 ,

$$\exp(\pi i J_2) \cdot |J, J_3\rangle = |J, -J_3\rangle, \text{ when } J \in \mathbb{Z} \text{ and } J_3 \neq 0. \quad (\text{A.14})$$

Using this property one can easily check the β -invariance of states in B -type. The index over gravitons in $AdS_4 \times S^7/\langle\alpha, \beta\rangle = AdS_4 \times (S^7/\mathbb{Z}_m)/\mathbb{Z}_2$ can be written as

$$I_{(S^7/\mathbb{Z}_m)/\mathbb{Z}_2:sp}(x, y) = I_{(S^7/\mathbb{Z}_m)/\mathbb{Z}_2:sp}^{(0)}(x) + \sum_{n>0} I_{S^7;sp}^{(mn)}(x) y^{mn}. \quad (\text{A.15})$$

Since β flip sign of J_3 quantum number, J_3 is no longer a good quantum number in $S^7/\langle\alpha, \beta\rangle$. But $|J_3|$ is still a good quantum number and y is chemical potential for the quantum number. The quantum number $|J_3|$ correspond to monopole charge $\sum_i |n_i|$ in $\mathcal{N} = 4$ $O(2N), USp(2N), SO(2N+1)$ SYMs which have M -theory on $AdS_4 \times S^7/\langle\alpha, \beta\rangle$ as gravity dual in infrared limit. First term $I_{S^7/\langle\alpha, \beta\rangle;sp}^{(0)}$ collect index contribution from gravitons in A-type, that is

$$\begin{aligned} I_{(S^7/\mathbb{Z}_m)/\mathbb{Z}_2:sp}^{(0)}(x) &= \sum_{J, J'; (-1)^{J+2J'}=1} (2J' + 1) I_{S^7}^{(J, J')}(x), \\ &= \frac{x(3 + 2x^{1/2} + 2x - 2x^{5/2} - x^3)}{(1 - x^2)^2}. \end{aligned} \quad (\text{A.16})$$

The second term in (A.15) comes from gravitons in B-type and $I_{S^7;sp}^{(n)}(x)$ here is same as that in (A.7).

B. Large N index on $\mathcal{N} = 4$ SYMs

B.1 $U(N) \oplus (\mathfrak{m} \text{ fundamentals})$

Using the general superconformal index formula in section 3.1, it straightforward to write down the superconformal index formula for the SYM theory.

$$\begin{aligned} I_{U(N):m}(x, y) &= \sum_{\{s=\{n_i\}\}} \frac{1}{(\text{sym})} y^{m \sum_i n_i} x^{\epsilon_0} \int \prod_{i=1}^N d\lambda_i \exp \left[\sum \frac{1}{n} f_{U(N) \oplus m}(x^n, e^{in\lambda_i}) \right], \\ f_{U(N):m}(x, e^{i\lambda_i}) &= 2 \sum_{i,j=1}^N \frac{x^{1/2}}{1+x} e^{i(\lambda_i - \lambda_j)} x^{|n_i - n_j|} + m \sum_{i=1}^N \frac{x^{1/2}}{1+x} (e^{i\lambda_i} + e^{-\lambda_i}) x^{|n_i|} \\ &\quad - \sum_{i \neq j} e^{i(\lambda_i - \lambda_j)} x^{|n_i - n_j|}, \quad \epsilon_0 = \frac{m}{2} \sum_i |n_i|. \end{aligned} \quad (\text{B.1})$$

Here the chemical potential y for $U(1)_{diag} \subset U(N)$ monopole charge is introduced. In the gravity side, the monopole charge quantum number can be identified with a generator of \mathbb{Z}_m in $AdS_4 \times S^7/\mathbb{Z}_m$.

To take large N limit on the index, we introduce distribution function $\rho(\theta)$ as

$$\rho(\theta) = \sum_{i=N_1+1}^N \delta(\lambda_i - \theta). \quad (\text{B.2})$$

Here N_1 denote the number of non-zero monopole fluxes of $U(1)^{N_1} \subset U(N)$,

$$s = \text{diag}\{n_1, \dots, n_{N_1}, 0, \dots, 0\}, \quad n_i \neq 0. \quad (\text{B.3})$$

Since θ is periodic variable, it's convenient to introduce Fourier transformation coefficients of $\rho(\theta)$, denoted by $\{\rho_n\}$

$$\rho_n = \int d\theta \rho(\theta) e^{in\theta} = \sum_{i=N_1+1}^N e^{in\lambda_i}. \quad (\text{B.4})$$

In the large N limit, holonomy integrals can be replaced by functional integral of distribution function $\rho(\theta)$

$$\int \prod_{i=1}^N d\lambda_i \rightarrow \int \prod_{i=1}^{N_1} d\lambda_i \int D[\rho(\theta)] = \int \prod_{i=1}^{N_1} d\lambda_i \int \prod_{n=1}^{\infty} d\rho_n d\rho_{-n}. \quad (\text{B.5})$$

Using the variables $\{\rho_n\}$, the index (B.1) for given monopole charges (B.3) can be written as follows in the large N limit,

$$I_{U(\infty):m}(x, y) = I_{N_1}(x, y) \int \prod_{n=1}^{\infty} d^2 x_n \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} \left(-\frac{1}{2} x_n^T M(\cdot^n) x_n + V^T(\cdot^n) x_n \right) \right],$$

where,

$$x_n := (\rho_n, \rho_{-n})^T, \quad M = \begin{pmatrix} 0 & 1 - 2h(x) \\ 1 - 2h(x) & 0 \end{pmatrix}, \quad h(x) := \frac{x^{1/2}}{1+x}$$

$$V = \left(mh(x) + [2h(x) - 1] \sum_{i=1}^{N_1} x^{|n_i|} e^{-i\lambda_i}, \quad mh(x) + [2h(x) - 1] \sum_{i=1}^{N_1} x^{|n_i|} e^{i\lambda_i} \right)^T. \quad (\text{B.6})$$

$I_{N_1}(x, y)$ denote the index (B.1) with $N = N_1$ and monopole charge $s_{N_1} = (n_1, \dots, n_{N_1})$. Performing the Gaussian integrations in \vec{x}_n (ignoring x -independent factors),

$$\begin{aligned} & \int \prod_{n=1}^{\infty} d^2 x_n \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(-\frac{1}{2} x_n^T M(\cdot^n) x_n + V^T(\cdot^n) x_n \right) \right), \\ &= \prod_{n=1}^{\infty} \frac{1}{\sqrt{\det M(\cdot^n)}} \exp \left(\sum_{n=1}^{\infty} \frac{1}{2} V(\cdot^n)^T M(\cdot^n)^{-1} V(\cdot^n) \right), \\ &= \left(\frac{1}{\prod_{n=1}^{\infty} [1 - 2h(x^n)]} \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{m^2 h^2(x^n)}{1 - 2h(x^n)} \right] \right) \times \\ & \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(-\sum_{i=1}^{N_1} mh(x^n) x^{n|n_i|} (e^{in\lambda_i} + e^{-in\lambda_i}) + \sum_{i,j=1}^{N_1} (1 - 2h(x^n)) x^{n(|n_i|+|n_j|)} e^{in(\lambda_i - \lambda_j)} \right) \right). \end{aligned} \quad (\text{B.7})$$

When all monopole charges are zero, $N_1 = 0$ and the second factor in the above is 1 and the first factor give the large N index. Thus the first factor can be considered as monopole zero index, which can be written as follows

$$I_{U(\infty):m}^{(0)}(x) = \frac{1}{\prod_{n=1}^{\infty} (1 - 2h(x^n))} \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{m^2 h^2(x^n)}{1 - 2h(x^n)} \right) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} I_{U(\infty):m:sp}^{(0)}(x^n) \right),$$

with $I_{U(\infty):m:sp}^{(0)}(x) = \frac{2(x^{1/2} + x - x^{5/2})}{1 - x^{1/2} - x^2 + x^{5/2}} + (m^2 - 1) \frac{x}{(1+x)(1-x^{1/2})^2}.$ (B.8)

Summarizing, the large N index is given by

$$\begin{aligned} I_{U(\infty):m}(x, y) &= I_{U(\infty):m}^{(0)}(x) I'_{U(\infty):m}(x, y), \text{ where} \\ I'_{U(\infty):m}(x, y) &= I_{N_1}(x, y) \times (\text{second factor in eq. (B.7)}) , \\ &= \frac{1}{(\text{sym})} x^{\epsilon_0} y^m \sum_{i=1}^{N_1} n_i \int \prod_{i=1}^{N_1} d\lambda_i \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} f'_{U(\infty):m}(x^n, e^{in\lambda_i}) \right), \text{ with} \\ f'_{U(\infty):m}(x, y) &= \sum_{i,j=1}^{N_1} \left(2h(x)(x^{|n_i-n_j|} - x^{|n_i|+|n_j|}) - ((1-\delta_{ij})x^{|n_i-n_j|} - x^{|n_i|+|n_j|}) \right) e^{i(\lambda_i-\lambda_j)}. \end{aligned}$$
 (B.9)

As with the large N index for ABJM theory [8], the large N index for the SYM exhibit following factorization properties

$$\sum_{\{n_i\}} I'_{U(\infty):m}(x, y) = \left(\sum_{\{n_i>0\}} I'_{U(\infty):m}(x, y) \right) \left(\sum_{\{n_i<0\}} I'_{U(\infty):m}(x, y) \right) := I_{U(\infty):m}^{(+)}(x, y) I_{U(\infty):m}^{(-)}(x, y). \quad (\text{B.10})$$

From the large N index formula (B.9), one can easily find following relation between positive/negative monopole charge index $I_{U(\infty):m}^{(\pm)}$

$$I_{U(\infty):m}^{(+)}(x, y) = I_{U(\infty):m}^{(-)}(x, 1/y). \quad (\text{B.11})$$

Note that m (number of fundamental hypermultiplet) dependence in the large N index $I'_{U(\infty):m}$ only appears as pre-factor in front of holonomy integral,

$$x^{\epsilon_0} y^m \sum n_i = x^{\frac{m}{2} \sum |n_i|} y^{m \sum n_i}. \quad (\text{B.12})$$

Due to this simple dependence of large N index on m , one can easily relate $I_{U(\infty):m}^{(\pm)}(x, y)$ for general m to that for $m = 1$. Let $I_{U(\infty):m=1}^{(+)}(x, y)$ be written as

$$\begin{aligned} I_{U(\infty):m=1}^{(+)}(x, y) &= \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} I_{U(\infty):m=1:sp}^{(+)}(x^n, y^n) \right], \\ I_{U(\infty):m=1:sp}^{(+)}(x, y) &= \sum_{n=1}^{\infty} I_{U(\infty):m=1:sp}^{(n)}(x) y^n. \end{aligned} \quad (\text{B.13})$$

Then for general m , the large N index $I_{U(\infty):m}^{(+)}(x, y)$ is given by

$$I_{U(\infty):m}^{(+)}(x, y) = \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} I_{U(\infty):m:sp}^{(+)}(x, {}^n y^n) \right], \text{ with}$$

$$I_{U(\infty):m:sp}^{(+)}(x, y) = \sum_{n=1}^{\infty} I_{U(\infty):m=1:sp}^{(n)}(x) x^{\frac{n(m-1)}{2}} y^{mn}. \quad (\text{B.14})$$

Negative monopole charge index $I_{U(\infty):m}^{(-)}$ also can be related to the case when $m = 1$ using the relation eq. B.11.

B.2 $O(2N)$, $Sp(2N)$, $SO(2N+1)$

To write down the superconformal index for $SO(2N)/Sp(2N)/SO(2N+1)$ gauge group, first we need to know weights and root of these gauge group. Summarizing the results

$$\begin{aligned} SO(2N) \quad & \rho \in 2N \quad \{\pm e_i\}_{i=1, \dots, N} \\ & \alpha \in G \quad \{(e_i - e_j)\}_{i,j=1, \dots, N}, \{\pm(e_i + e_j)\}_{i < j=1, \dots, N}, \\ \\ Sp(2N) \quad & \rho \in 2N \quad \{\pm e_i\}_{i=1, \dots, N} \\ & \alpha \in G \quad \{(e_i - e_j)\}_{i,j=1, \dots, N}, \{\pm(e_i + e_j)\}_{i < j=1, \dots, N}, \{\pm 2e_i\}_{i=1, \dots, N} \\ \\ SO(2N+1) \quad & \rho \in 2N+1 \quad \{\pm e_i, 0\}_{i=1, \dots, N} \\ & \alpha \in G \quad \{(e_i - e_j)\}_{i,j=1, \dots, N}, \{\pm(e_i + e_j)\}_{i < j=1, \dots, N}, \{\pm e_i\}_{i=1, \dots, N}. \end{aligned}$$

In large N limit, $O(2N)$ SYM with m fundamental hypermultiplets can't be distinguished from the $SO(2N)$ SYM with same matter contents. Thus we will consider the large N limit on $SO(2N)$ SYM. Using the general index formula in sec 3.1 and weight and roots of $SO(2N)$, the superconformal index can be written as

$$\begin{aligned} I_{SO(2N):m}(x, y) &= \sum_{\{s=\{n_i>0\}\}} \frac{1}{(\text{sym})} y^{m \sum_i |n_i|} x^{\epsilon_0} \int \prod_{i=1}^N d\lambda_i \exp \left[\sum \frac{1}{n} f_{SO(2N):m}(x^n, e^{in\lambda_i}) \right], \\ f_{SO(2N):m}(x, e^{i\lambda_i}) &= \sum_{i,j=1}^N 2h(x) e^{i(\lambda_i - \lambda_j)} x^{|n_i - n_j|} + \sum_{i < j}^N 2h(x) (e^{i(\lambda_i + \lambda_j)} + e^{-i(\lambda_i + \lambda_j)}) x^{|n_i + n_j|}, \quad h(x) := \frac{x^{1/2}}{1+x}. \\ &+ \sum_{i=1}^N 2mh(x) (e^{i\lambda_i} + e^{-i\lambda_i}) x^{|n_i|} - \sum_{i,j=1}^N e^{i(\lambda_i - \lambda_j)} x^{|n_i - n_j|} - \sum_{i < j}^N (e^{i(\lambda_i + \lambda_j)} + e^{-i(\lambda_i + \lambda_j)}) x^{|n_i + n_j|}. \end{aligned} \quad (\text{B.15})$$

We introduce the chemical potential y for monopole charge $\sum_i |n_i|$. Using the $(\mathbb{Z}_2)^N$ symmetry in Weyl group of $O(2N)$ we can take all monopole charges to be positive. Following the same procedure in the $U(N)$ SYM case, the large N index of the $O(2N)$ SYM becomes

$$\begin{aligned}
& I_{O(\infty):m}(x, y) \\
&= I_{O(N_1):m}(x, y) \int \prod_{n=1}^{\infty} d\chi_n \exp \left(- \sum_{n=1}^{\infty} \frac{1}{2n} ((1 - 2h(x^n))\chi_n^2 - (1 - 2h(x^n))\chi_{2n} \right. \\
&\quad \left. - 4mh(x^n)\chi_n - 2\chi_n v_n) \right), \quad v_n := (2h(x^n) - 1) \sum_{i=1}^{N_1} x^{n|n_i|} (e^{in\lambda_i} + e^{-in\lambda_i}). \\
&= I_{O(N_1):m}(x, y) \prod_{k=1}^{\infty} \frac{\exp \left[\frac{(2mh(x^{2k-1}) + v_{2k-1})^2}{2(2k-1)(1-2h(x^{2k-1}))} + \frac{(1-2h(x^k) + 2mh(x^{2k}) + v_{2k})^2}{4k(1-2h(x^{2k}))} \right]}{\sqrt{1-2h(x^{2k-1})}\sqrt{1-2h(x^{2k})}}, \\
&:= I_{O(\infty):m}^{(0)}(x) I'_{O(\infty):m}(x, y). \tag{B.16}
\end{aligned}$$

Here we introduce variables $\chi_n = \sum_{i=1}^{N_1} (e^{i\lambda_i n} + e^{-i\lambda_i n})$, with monopole charge $s = \{n_1, \dots, n_{N_1}, 0, \dots, 0\}$. Monopole charge zero sector index $I_{O(\infty):m}^{(0)}(x)$ is given by

$$\begin{aligned}
& I_{O(\infty):m}^{(0)}(x) \\
&= \frac{1}{\prod_{n=1}^{\infty} \sqrt{1-2h(x^n)}} \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2m^2 h^2}{1-2h}(x^n) + \frac{mh(x^{2n})(1-2h(x^n)) + (h(x^n) - \frac{1}{2})^2}{1-2h(x^{2n})} \right) \right) \\
&= \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} I_{O(\infty):m:sp}^{(0)}(x^n) \right], \text{ where} \\
& I_{O(\infty):m:sp}^{(0)}(x) = \frac{x(3 + 2x^{1/2} + 2x - 2x^{5/2} - x^3)}{(1-x^2)^2} \\
&\quad + (2m^2 + m) \frac{x}{(1-x)^2} + (2m^2 - m - 1) \frac{2x^{3/2}}{(1-x)^2(1+x)} \tag{B.17}
\end{aligned}$$

The remaining part in large N index $I'_{O(\infty):m}(x, y)$ in eq. (B.16) is given by

$$\begin{aligned}
& I'_{O(\infty):m}(x, y) \\
&= I_{O(N_1):m}(x, y) \prod_{k=1}^{\infty} \exp \left(\frac{4mh(x^{2k-1})v_{2k-1} + v_{2k-1}^2}{2(2k-1)(1-2h(x^{2k-1}))} + \frac{(4mh(x^{2k}) + 2(1-2h(x^k)))v_{2k} + v_{2k}^2}{4k(1-2h(x^{2k}))} \right) \\
&= \frac{1}{(\text{sym})} y^{m \sum_{i=1}^{N_1} |n_i|} x^{\epsilon_0} \prod_{i=1}^{N_1} d\lambda_i \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} f'_{O(\infty):m}(x^n, e^{in\lambda_i}) \right), \text{ with} \\
& f'_{O(\infty):m} = \sum_{i,j=1}^{N_1} \left(2h(x)(x^{|n_i-n_j|} - x^{|n_i|+|n_j|}) - ((1-\delta_{ij})x^{|n_i-n_j|} - x^{|n_i|+|n_j|}) \right) e^{i(\lambda_i-\lambda_j)}. \tag{B.18}
\end{aligned}$$

Note that $I'_{O(\infty):m}(x, y)$ is same with positive monopole charge part $I_{U(\infty):m}^{(+)}(x, y)$ for $U(N)$ SYM case (B.9). This is consistent with the fact that non-zero KK momentum graviton index on $AdS_4 \times (S^7/\mathbb{Z}_m)/\mathbb{Z}_2$ is equal to positive KK momentum graviton index on $AdS_4 \times S^7$, which is explicitly shown in eq. (A.15).

Using the same technique, one can calculate the large N index for $SO(2N+1)$, $Sp(2N)$ SYM with m fundamental hypermultiplets. For $SO(2N+1)$ SYM, the large N index is same as $O(2N)$ SYM. For $Sp(2N)$ SYM, the non-zero monopole charge sector of large N index coincide with $O(2N)$ SYM case, that is

$$I'_{Sp(\infty):m}(x, y) = I'_{O(\infty):m}(x, y) . \quad (\text{B.19})$$

But in zero monopole charge sector large N index is somewhat different, the result is

$$\begin{aligned} I_{Sp(\infty):m:sp}^{(0)}(x) &= I_{O(\infty):-m:sp}^{(0)}(x) \\ &= \frac{x(3 + 2x^{1/2} + 2x - 2x^{5/2} - x^3)}{(1-x^2)^2} + (2m^2 - m) \frac{x}{(1-x)^2} + (2m^2 + m - 1) \frac{2x^{3/2}}{(1-x)^2(1+x)} . \end{aligned} \quad (\text{B.20})$$

C. Partition Function

Let us consider the brane configurations given in fig. 4. In (a), there are N D3-branes diagramed by the circle, n NS5-branes, and $m = m_1 + m_2 + \dots + m_L$ D5-branes, m_i D5-branes between the i th and $i+1$ th NS5-branes. As in section 5, we take T-duality to obtain SCS theories from brane configuration in (b).

The fig. 4 (a) corresponds to 3-d, $\mathcal{N} = 4$, $U(N)^n$ SYM with n bi-fundamental, and m fundamental hyper-multiplets in following representation,

$$\begin{aligned} &(N, \bar{N}, 1, \dots, 1) \oplus (1, N, \bar{N}, 1, \dots, 1) \oplus \dots \oplus (\bar{N}, 1, \dots, 1, N) \\ &m_1(N, 1, \dots, 1) \oplus m_2(1, N, 1, \dots, 1) \oplus \dots \oplus m_L(\underbrace{1, \dots, 1}_{L-1}, N, \underbrace{1, \dots, 1}_{n-L}) \end{aligned}$$

The IR limit of fig. 4 (b) gives $\mathcal{N} = 4$, $U(N)^{n+m}$ CS with $(n+m)$ bi-fundamental hyper-multiplets in

$$(N, \bar{N}, 1, \dots, 1) \oplus (1, N, \bar{N}, 1, \dots, 1) \dots \oplus (\bar{N}, 1, \dots, 1, N),$$

The Chern-Simons level can be one of $1, -1, 0$, depending on whether D3-branes are between NS'/D5, D5/NS', (D5/D5 or NS'/NS'). The Chern-Simons levels from the brane configuration (b) can be denoted as a following $(n+m)$ -vector,

$$\vec{k} = (1, \underbrace{0, \dots, 0}_{m_1-1}, -1, 1, \underbrace{0, \dots, 0}_{m_2-1}, -1, \dots, \underbrace{0, \dots, 0}_{m_L-1}, -1, \underbrace{0, \dots, 0}_{n-L}) \quad (\text{C.1})$$

Note that for the same number of 5-branes, the moduli spaces are same [27] though the corresponding theories can differ by the ordering of the 5-branes.

We will show that the partition function for two theories are same. The procedure is similar to the proof in [5] which shows the equivalence of the partition function of $U(N)_1 \times U(N)_{-1}$ ABJM and the 3-d $\mathcal{N} = 8$, $U(N)$ YM. However we will show the calculation explicitly to be self-contained. The partition function of $\mathcal{N} = 4$ SCS is given by

$$Z_{CS} = \frac{1}{(N!)^{n+m}} \int (d^N \sigma)^{n+m} \frac{\prod_{i < j} \sinh^2(\pi \sigma_{ij}^1) \cdots \sinh^2(\pi \sigma_{ij}^{n+m})}{\prod_{i,j} \cosh(\pi(\sigma_i^1 - \sigma_j^2)) \cdots \cosh(\pi(\sigma_i^{n+m} - \sigma_j^1))} e^{iS_0} \quad (\text{C.2})$$

where σ_i^I for $I = 1, \dots, n+m$, $i = 1, \dots, N$ are the moduli of the theory. S_0 is the classical action,

$$S_0 = \pi \sum_{i=1}^N [(\sigma_i^1)^2 - (\sigma_i^{m_1+1})^2 + \cdots - (\sigma_i^{m+L})^2] = \pi \sum_{i=1}^N \sum_{I=1}^{n+m} k_I (\sigma_i^I)^2$$

where k_I is the Chern-Simons level of I th gauge group, given in (C.1). Let us define a sequence $\{n\}$ as $(n_1, n_2, \dots, n_L) := (m_1+1, m_1+m_2+2, \dots, (m_1+m_2+\dots+m_L)+L)$ to denote the gauge groups with $k = -1$, and define $\{\tilde{n}\}$ as $(\tilde{n}_1, \dots, \tilde{n}_L) := (1, m_1+2, m_1+m_2+3, \dots, (m_1+\dots+m_{L-1})+L)$ to denote the gauge groups with $k = 1$. The classical action can be written as $S_0 = \pi \sum_{I=1}^L \sum_i ((\sigma_i^{\tilde{n}_I})^2 - (\sigma_i^{n_I})^2)$. Introduce permutations $\rho_I : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$, then the partition function becomes

$$Z_{CS} = \sum_{\rho_1, \dots, \rho_{n+m}} \frac{(-1)^{\rho_1 + \dots + \rho_{n+m}}}{(N!)^{n+m}} \int (d^N \sigma)^{n+m} \frac{e^{\pi i \sum_{I=1}^L ((\sigma^{\tilde{n}_I})^2 - (\sigma^{n_I})^2)}}{\prod_i \cosh(\pi(\sigma_i^1 - \sigma_{\rho_1(i)}^2)) \cdots \cosh(\pi(\sigma_i^{n+m} - \sigma_{\rho_{n+m}(i)}^1))}.$$

using (C.5). One can redefine variables $\sigma_{\rho_1(i)}^2 \rightarrow \sigma_i^2$, repeatedly for $\sigma_{\rho_{I-1}(i)}^I$, then for $\rho := \rho_1 + \rho_2 + \dots + \rho_{n+m}$, it becomes

$$Z_{CS} = \sum_{\rho} \frac{(-1)^{\rho}}{N!} \int (d^N \sigma)^{n+m} \frac{e^{\pi i \sum_{I=1}^L ((\sigma^{\tilde{n}_I})^2 - (\sigma^{n_I})^2)}}{\prod_i \cosh(\pi(\sigma_i^1 - \sigma_i^2)) \cosh(\pi(\sigma_i^2 - \sigma_i^3)) \cdots \cosh(\pi(\sigma_i^{n+m} - \sigma_{\rho(i)}^1))}.$$

Now use (C.6) to introduce new integral variables τ_i^I . Then integrate over the variables of the gauge groups with vanishing Chern-Simons levels using (C.7) to obtain

$$\begin{aligned} & \sum_{\rho} \frac{(-1)^{\rho}}{N!} \int (d^N \sigma \, d^N \tau)^{n+m} \frac{e^{\pi i \sum_i \sum_{I=1}^L ((\sigma_i^{n_I})^2 - (\sigma_i^{\tilde{n}_I})^2)} e^{2\pi i \sum_i \sum_I (\tau_i^I (\sigma_i^I - \sigma_i^{I+1}))}}{\prod_i \cosh(\pi \tau_i^1) \cosh(\pi \tau_i^2) \cdots \cosh(\pi \tau_i^{n+m})} \\ &= \sum_{\rho} \frac{(-1)^{\rho}}{N!} \int (d^N \sigma \, d^N \tau)^{2L} \frac{e^{\pi i \sum_{i,I} ((\sigma_i^{n_I})^2 - (\sigma_i^{\tilde{n}_I})^2)} e^{2\pi i (\sigma_i^{\tilde{n}_1} (\tau_i^{\tilde{n}_1} - \tau_{\rho(i)}^{n_L}) + \dots + \sigma_i^{\tilde{n}_L} (\tau_i^{\tilde{n}_L} - \tau_i^{n_{L-1}}) + \sigma_i^{n_L} (\tau_i^{n_L} - \tau_i^{\tilde{n}_L}))}}{\prod_i \cosh(\pi \tau_i^{n_1}) \cdots \cosh(\pi \tau_i^{n_{L-1}}) \cosh^{n-L+1}(\pi \tau_i^{n_L}) \prod_{i,I} \cosh^{m_I}(\pi \tau_i^{\tilde{n}_I})}. \end{aligned}$$

Doing the Gaussian integral results in

$$Z_{CS} = \sum_{\rho} \frac{(-1)^{\rho}}{N!} \int (d^N \tau)^{2L} \frac{e^{2\pi i \sum_i (\tau_i^{n_1} (\tau_i^{n_1} - \tau_{\rho(i)}^{n_L}) + \sum_{I=2}^L \tau_i^{n_I} (\tau_i^{n_I} - \tau_i^{n_{I-1}}))}}{\prod_i (\prod_{I=1}^{L-1} \cosh(\pi \tau_i^{n_I})) \cosh^{n-L+1}(\pi \tau_i^{n_L}) \prod_{I=1}^L \cosh^{m_I}(\pi \tau_i^{n_I})} . \quad (\text{C.3})$$

On the other hand, the partition function of the $\mathcal{N} = 4$, $U(N)^n$ YM is given by

$$Z_{YM} = \frac{1}{(N!)^n} \int (d^N \sigma)^n \frac{\prod_{i < j} \sinh^2(\pi \sigma_{ij}^1) \cdots \sinh^2(\pi \sigma_{ij}^n)}{\prod_{i,j} \cosh(\pi(\sigma_i^1 - \sigma_j^2)) \cdots \cosh(\pi(\sigma_i^n - \sigma_j^1)) \prod_{i=1}^N \prod_I \cosh^{m_I}(\pi \sigma_i^I)} \quad (\text{C.4})$$

One can now use (C.5) to rewrite $\frac{\prod_{i < j} \sinh(\pi \sigma_{ij}^I) \sinh(\pi \sigma_{ij}^{I+1})}{\prod_{i,j} \cosh(\pi(\sigma_i^I - \sigma_j^{I+1}))} = \sum_{\rho_I} (-1)^{\rho_I} \frac{1}{\cosh(\pi(\sigma_i^I - \sigma_{\rho_I(i)}^{I+1}))}$, then redefine variables repeatedly to get

$$Z_{YM} = \sum_{\rho} \frac{(-1)^{\rho}}{N!} \int (d^N \sigma)^n \frac{1}{\prod_i \left(\left[\prod_{I=1}^{n-1} \cosh(\pi(\sigma_i^I - \sigma_i^{I+1})) \right] \cosh(\pi(\sigma_i^n - \sigma_{\rho(i)}^1)) \prod_{I=1}^n \cosh^{m_I}(\pi \sigma_i^I) \right)} .$$

Now use (C.6) to introduce τ_i^I variables, then integrate out $\sigma_i^{L+1}, \dots, \sigma_i^n$ using (C.7) to obtain

$$Z_{YM} = \sum_{\rho} \frac{(-1)^{\rho}}{N!} \int (d^N \sigma)^L (d^N \tau)^L \frac{e^{2\pi i \sum_i (\sigma_i^1 (\tau_i^1 - \tau_{\rho(i)}^L) + \sigma_i^2 (\tau_i^2 - \tau_i^1) + \cdots + \sigma_i^L (\tau_i^L - \tau_i^{L-1}))}}{\prod_i \left(\left(\prod_{I=1}^{L-1} \cosh(\pi \tau_i^I) \right) \cosh^{n-L+1}(\pi \tau_i^L) \prod_{I=1}^L \cosh^{m_I}(\pi \sigma_i^I) \right)} .$$

It is equivalent to the integral in (C.3). This shows that for any 5-brane configuration of fig. 4 type, the partition functions of $\mathcal{N} = 4$ CS's and YM's are same

$$Z_{CS} = Z_{YM} .$$

C.1 Useful Formulas

In [5], the following identity is proved

$$\frac{\prod_{i < j} \sinh(x_i - x_j) \sinh(y_i - y_j)}{\prod_{i,j} \cosh(x_i - y_j)} = \sum_{\rho} (-1)^{\rho} \prod_i \frac{1}{\cosh(x_i - y_{\rho(i)})} , \quad (\text{C.5})$$

where $(-1)^{\rho}$ is defined to be $1(-1)$ for an even(odd) permutation ρ .

The following identities are useful forms of Fourier transform,

$$\frac{1}{\cosh(\pi \sigma)} = \int d\tau \frac{e^{2\pi i \tau \sigma}}{\cosh(\pi \tau)} , \quad (\text{C.6})$$

$$\delta(\tau^I - \tau^J) = \int d\sigma e^{2\pi i (\tau^I - \tau^J) \sigma} . \quad (\text{C.7})$$

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